CHAPTER 9

CHAPTER 9 Transformations

Chapter Outline

The final chapter of Geometry explores transformations. A transformation is a move, flip, or rotation of an image. First, we will look at different types of symmetry and then discuss the different types of transformations. Finally, we will compose transformations and look at tessellations.

9.1 Exploring Symmetry

Learning Objectives

- Learn about lines of symmetry.
- Discuss line and rotational symmetry.
- Learn about the center of symmetry.

Review Queue

a. Define symmetry in your own words.
b. Draw a regular hexagon. How many degrees does each angle have?
c. Draw all the diagonals in your hexagon. What is the measure of each central angle?
d. Plot the points $A(1,3), B(3,1), C(5,3)$, and $D(3,5)$. What kind of shape is this? Prove it using the distance formula and/or slope.

Know What? Symmetry exists all over nature. One example is a starfish, like the one below. Draw in the line(s) of symmetry, center of symmetry and the angle of rotation for this starfish.

Lines of Symmetry

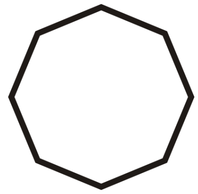
Line of Symmetry: A line that passes through a figure such that it splits the figure into two congruent halves.

Many figures have a line of symmetry, but some do not have any lines of symmetry. Figures can also have more than one line of symmetry.

Example 1: Find all lines of symmetry for the shapes below.

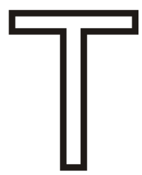
a)

b)



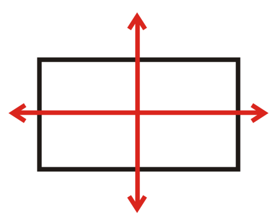
c)

d)

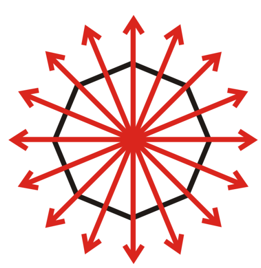


Solution: For each figure, draw lines that cut the figure in half perfectly. Figure a) has two lines of symmetry, b) has eight, c) has no lines of symmetry, and d) has one.

a)



b)



c)

d)

Figures a), b), and d) all have ***line symmetry***.

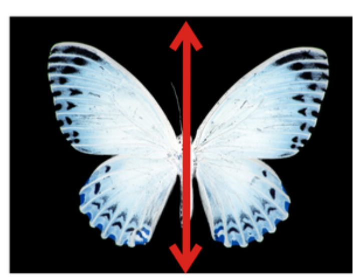
Line Symmetry: When a figure has one or more lines of symmetry.

Example 2: Do the figures below have line symmetry?

a)

b)

Solution: Yes, both of these figures have line symmetry. One line of symmetry is shown for the flower; however it has several more lines of symmetry. The butterfly only has one line of symmetry.



Rotational Symmetry

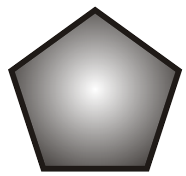
Rotational Symmetry: When a figure can be rotated (less that 360°) and it looks the same way it did before the rotation.

Center of Rotation: The point at which the figure is rotated around such that the rotational symmetry holds. Typically, the center of rotation is the center of the figure.

Along with rotational symmetry and a center of rotation, figures will have an ***angle of rotation***. The angle of rotation, tells us how many degrees we can rotate a figure so that it still looks the same.

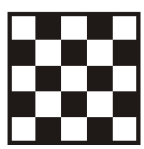
Example 3: Determine if each figure below has rotational symmetry. If it does, determine the angle of rotation.

a)



b)

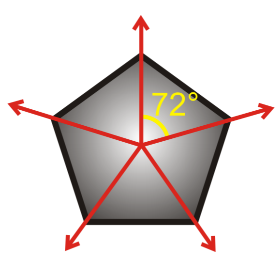
c)



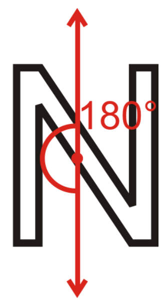
Solution:

a) The regular pentagon can be rotated 5 times so that each vertex is at the top. This means the angle of rotation is $\frac{360^\circ}{5} = 72^\circ$.

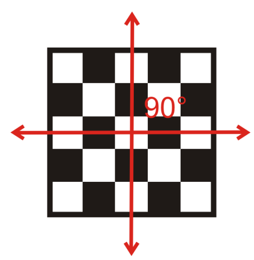
The pentagon can be rotated $72^\circ, 144^\circ, 216^\circ$, and 288° so that it still looks the same.



b) The "*N*" can be rotated twice, 180°, so that it still looks the same.

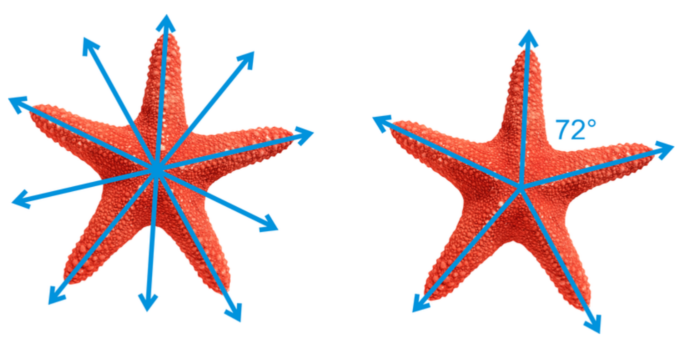


c) The checkerboard can be rotated 4 times so that the angle of rotation is $\frac{360^\circ}{4} = 90^\circ$. It can be rotated 180° and 270° as well. The final rotation is always 360° to get the figure back to its original position.



In general, ***if a shape can be rotated n times, the angle of rotation is*** $\frac{360^\circ}{n}$. Then, multiply the angle of rotation by 1, 2, 3..., and n to find the additional angles of rotation.

Know What? Revisited The starfish has 5 lines of symmetry and has rotational symmetry of 72°. Therefore, the starfish can be rotated $72^\circ, 144^\circ, 216^\circ$, and 288° and it will still look the same. The center of rotation is the center of the starfish.



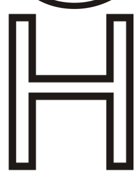
Review Questions

Determine if the following questions are ALWAYS true, SOMETIMES true, or NEVER true.

1. Right triangles have line symmetry.
2. Isosceles triangles have line symmetry.
3. Every rectangle has line symmetry.
4. Every rectangle has exactly two lines of symmetry.
5. Every parallelogram has line symmetry.
6. Every square has exactly two lines of symmetry.
7. Every regular polygon has three lines of symmetry.
8. Every sector of a circle has a line of symmetry.

9. Every parallelogram has rotational symmetry.
10. A rectangle has $90^\circ, 180^\circ$, and 270° angles of rotation.
11. Draw a quadrilateral that has two pairs of congruent sides and exactly one line of symmetry.
12. Draw a figure with infinitely many lines of symmetry.
13. Draw a figure that has one line of symmetry and no rotational symmetry.
14. Fill in the blank: A regular polygon with n sides has ______ lines of symmetry.

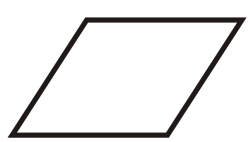
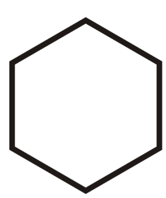
Find all lines of symmetry for the letters below.

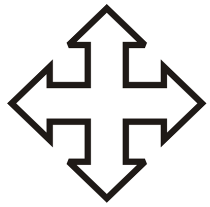
15. 
16. 
17. 
18. 
19. 
20. Do any of the letters above have rotational symmetry? If so, which one(s) and what are the angle(s) of rotation?

Determine if the words below have line symmetry or rotational symmetry.

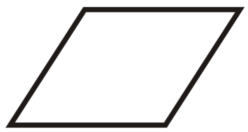
21. OHIO
22. **MOW**
23. **WOW**
24. **KICK**
25. **pod**

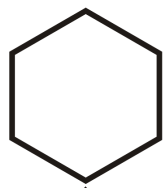
Trace each figure and then draw in all lines of symmetry.

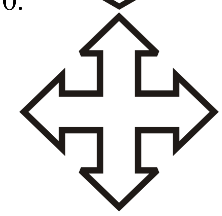
26. 
27. 

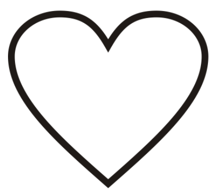
28. 

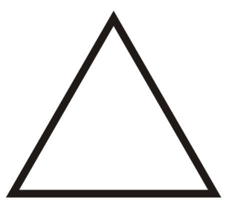
Find the angle(s) of rotation for each figure below.

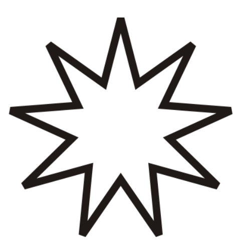
29. 

30. 

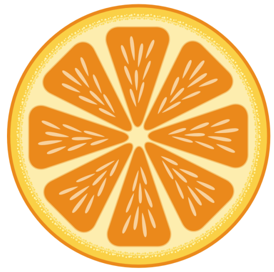
31. 

32. 

33. 

34. 

Determine if the figures below have line symmetry or rotational symmetry. Identify all lines of symmetry and all angles of rotation.

35. 

36. 

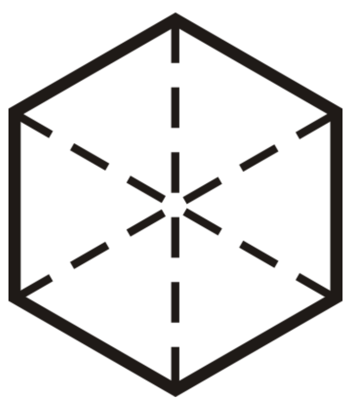
37. 

Review Queue Answers

1. Where one side of an object matches the other side; answers will vary.

2 and 3. each angle has $\frac{(n-1)180^\circ}{n} = \frac{5(180^\circ)}{6} = 120^\circ$

each central angle has $\frac{360^\circ}{6} = 60^\circ$



4. The figure is a square.

9.2 Translations and Vectors

Learning Objectives

- Graph a point, line, or figure and translate it x and y units.
- Write a translation rule.
- Use vector notation.

Review Queue

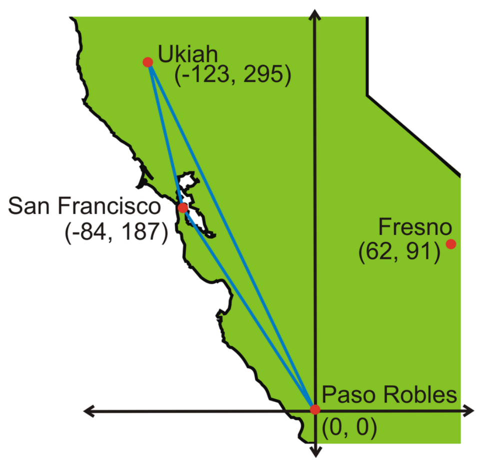
a. Find the equation of the line that contains (9, -1) and (5, 7).
b. What type of quadrilateral is formed by $A(1,-1), B(3,0), C(5,-5)$ and $D(-3,0)$?
c. Find the equation of the line parallel to #1 that passes through (4, -3).
d. Find the equation of the line perpendicular to #1 that passes through (4, -3).

Know What? Lucy currently lives in San Francisco, S, and her parents live in Paso Robles, P. She will be moving to Ukiah, U, in a few weeks. All measurements are in miles. Find:

a) The component form of $\overrightarrow{PS}, \overrightarrow{SU}$ and $\overrightarrow{PU}$.

b) Lucy's parents are considering moving to Fresno, F. Find the component form of $\overrightarrow{PF}$ and $\overrightarrow{UF}$.

c) Is Ukiah or Paso Robles closer to Fresno?



Transformations

Recall from Lesson 7.6, we learned about dilations, which is a type of transformation. Now, we are going to continue learning about other types of transformations. All of the transformations in this chapter are rigid transformations.

Transformation: An operation that moves, flips, or changes a figure to create a new figure.

Rigid Transformation: A transformation that preserves size and shape.

The rigid transformations are: translations, reflections, and rotations. The new figure created by a transformation is called the ***image***. The original figure is called the ***preimage***. Another word for a rigid transformation is an ***isometry***. Rigid transformations are also called congruence transformations.

Also in Lesson 7.6, we learned how to label an image. If the preimage is A, then the image would be labeled A', said "a prime." If there is an image of A', that would be labeled A'', said "a double prime."

Translations

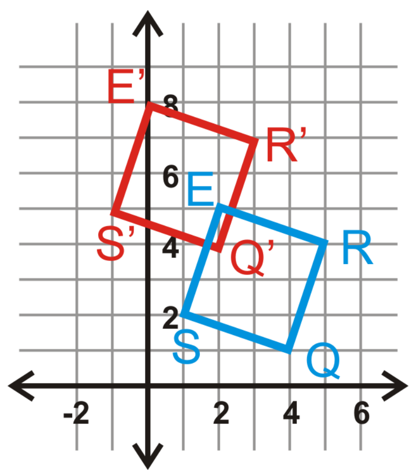
The first of the rigid transformations is a translation.

Translation: A transformation that moves every point in a figure the same distance in the same direction.

In the coordinate plane, we say that a translation moves a figure x units and y units.

Example 1: Graph square $S(1,2), Q(4,1), R(5,4)$ and $E(2,5)$. Find the image after the translation $(x,y) \rightarrow (x-2, y+3)$. Then, graph and label the image.

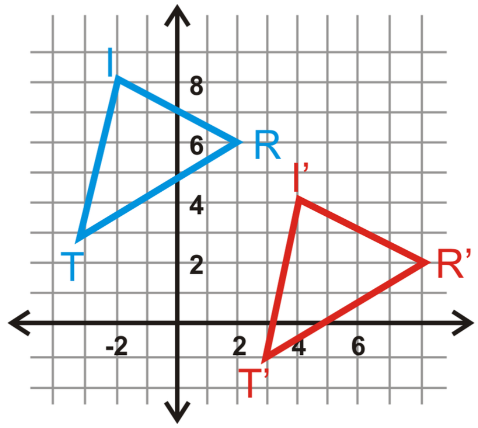
Solution: The translation notation tells us that we are going to move the square to the left 2 and up 3.



$$
\begin{aligned}
(x,y) &\rightarrow (x-2, y+3) \\
S(1,2) &\rightarrow S'(-1,5) \\
Q(4,1) &\rightarrow Q'(2,4) \\
R(5,4) &\rightarrow R'(3,7) \\
E(2,5) &\rightarrow E'(0,8)
\end{aligned}
$$

Example 2: Find the translation rule for $\triangle TRI$ to $\triangle T'R'I'$.

Solution: Look at the movement from T to T'. T is (-3, 3) and T' is (3, -1). The change in x is 6 units to the right and the change in y is 4 units down. Therefore, the translation rule is $(x,y) \rightarrow (x+6, y-4)$.



From both of these examples, we see that a translation preserves congruence. Therefore, ***a translation is an isometry***. We can show that each pair of figures is congruent by using the distance formula.

Example 3: Show $\triangle TRI \cong \triangle T'R'I'$ from Example 2.

Solution: Use the distance formula to find all the lengths of the sides of the two triangles.

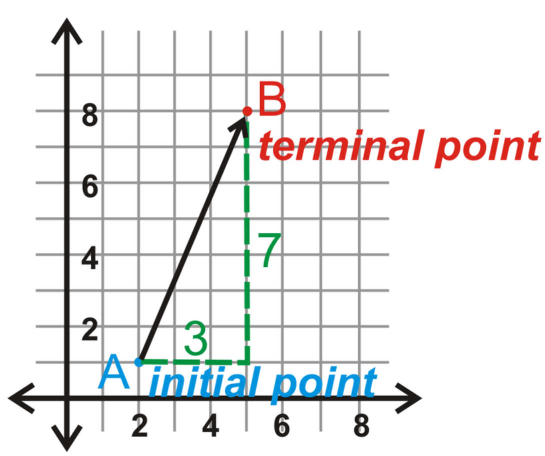
$\underline{\triangle TRI}$	$\underline{\triangle T'R'I'}$
$TR = \sqrt{(-3-2)^2+(3-6)^2} = \sqrt{34}$	$T'R' = \sqrt{(3-8)^2+(-1-2)^2} = \sqrt{34}$
$RI = \sqrt{(2-(-2))^2+(6-8)^2} = \sqrt{20}$	$R'I' = \sqrt{(8-4)^2+(2-4)^2} = \sqrt{20}$
$TI = \sqrt{(-3-(-2))^2+(3-8)^2} = \sqrt{26}$	$T'I' = \sqrt{(3-4)^2+(-1-4)^2} = \sqrt{26}$

Vectors

Another way to write a translation rule is to use vectors.

Vector: A quantity that has direction and size.

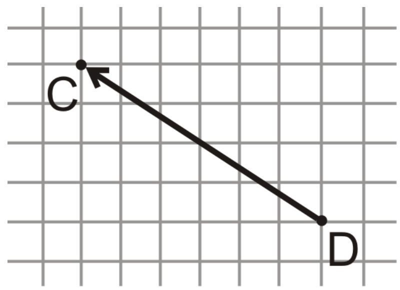
In the graph below, the line from A to B, or the distance traveled, is the vector. This vector would be labeled $\overrightarrow{AB}$ because A is the ***initial point*** and B is the ***terminal point***. The terminal point always has the arrow pointing towards it and has the half-arrow over it in the label.



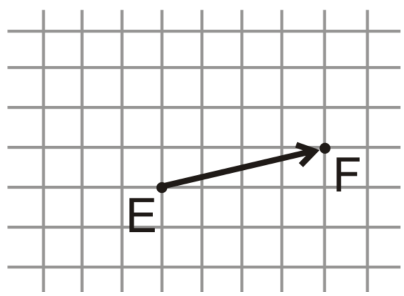
The ***component form*** of $\overrightarrow{AB}$ combines the horizontal distance traveled and the vertical distance traveled. We write the component form of $\overrightarrow{AB}$ as $\langle 3,7\rangle$ because $\overrightarrow{AB}$ travels 3 units to the right and 7 units up. Notice the brackets are pointed, $\langle 3,7\rangle$, not curved.

Example 4: Name the vector and write its component form.

a)



b)

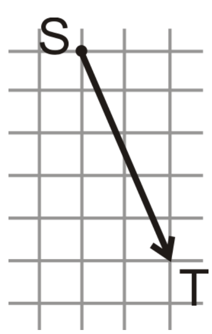


Solution:

a) The vector is $\overrightarrow{DC}$. From the initial point D to terminal point C, you would move 6 units to the left and 4 units up. The component form of $\overrightarrow{DC}$ is $\langle -6,4\rangle$.

b) The vector is $\overrightarrow{EF}$. The component form of $\overrightarrow{EF}$ is $\langle 4,1\rangle$.

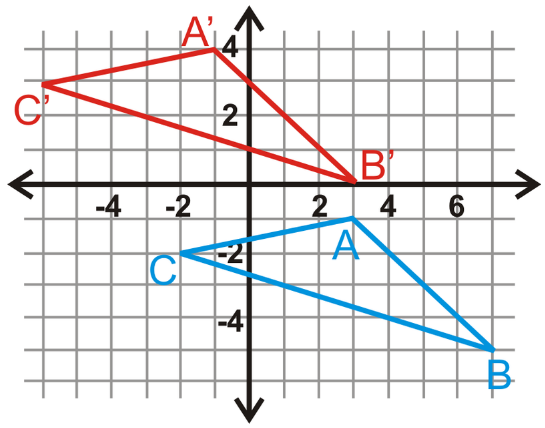
Example 5: Draw the vector $\overrightarrow{ST}$ with component form $\langle 2,-5\rangle$.



Solution: The graph above is the vector $\overrightarrow{ST}$. From the initial point S it moves down 5 units and to the right 2 units.

The positive and negative components of a vector always correlate with the positive and negative parts of the coordinate plane. We can also use vectors to translate an image.

Example 6: Triangle $\triangle ABC$ has coordinates $A(3,-1), B(7,-5)$ and $C(-2,-2)$. Translate $\triangle ABC$ using the vector $\langle -4,5\rangle$. Determine the coordinates of $\triangle A'B'C'$.



Solution: It would be helpful to graph $\triangle ABC$. To translate $\triangle ABC$, add each component of the vector to each point to find $\triangle A'B'C'$.

$$A(3,-1)+\langle -4,5\rangle = A'(-1,4)$$
$$B(7,-5)+\langle -4,5\rangle = B'(3,0)$$
$$C(-2,-2)+\langle -4,5\rangle = C'(-6,3)$$

Example 7: Write the translation rule for the vector translation from Example 6.

Solution: To write $\langle -4,5\rangle$ as a translation rule, it would be $(x,y) \rightarrow (x-4, y+5)$.

Know What? Revisited

a) $\overrightarrow{PS} = \langle -84, 187\rangle, \overrightarrow{SU} = \langle -39, 108\rangle, \overrightarrow{PU} = \langle -123, 295\rangle$

b) $\overrightarrow{PF} = \langle 62, 91\rangle, \overrightarrow{UF} = \langle 185, -204\rangle$

c) You can plug the vector components into the Pythagorean Theorem to find the distances. Paso Robles is closer to Fresno than Ukiah.

$$UF = \sqrt{185^2 + (-204)^2} \cong 275.4 \ miles, PF = \sqrt{62^2 + 91^2} \cong 110.1 \ miles$$

Review Questions

1. What is the difference between a vector and a ray?

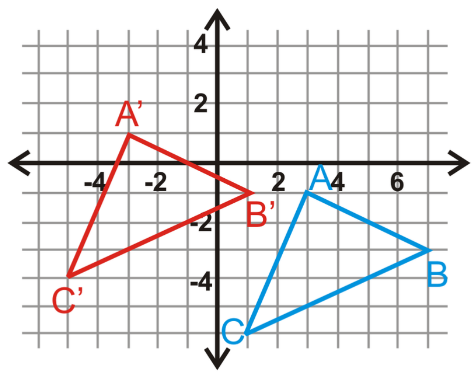
Use the translation $(x,y) \rightarrow (x+5, y-9)$ for questions 2-8.

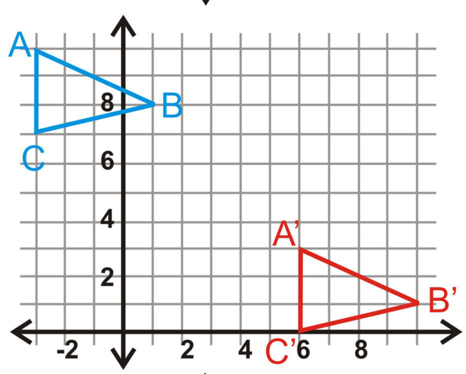
2. What is the image of $A(-6,3)$?
3. What is the image of $B(4,8)$?
4. What is the preimage of $C'(5,-3)$?
5. What is the image of A'?
6. What is the preimage of $D'(12,7)$?
7. What is the image of A''?
8. Plot A, A', A'', and A''' from the questions above. What do you notice? Write a conjecture.

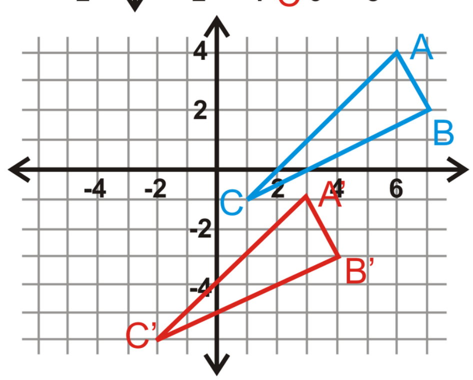
The vertices of $\triangle ABC$ are $A(-6,-7), B(-3,-10)$ and $C(-5,2)$. Find the vertices of $\triangle A'B'C'$, given the translation rules below.

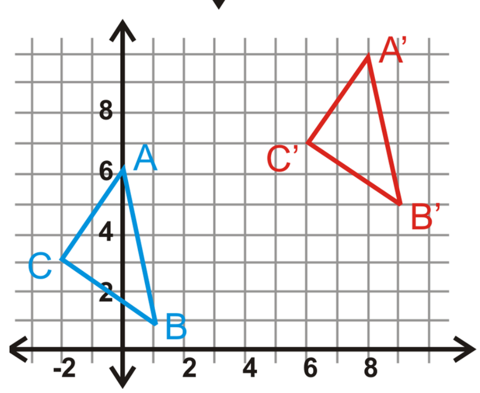
9. $(x,y) \rightarrow (x-2, y-7)$
10. $(x,y) \rightarrow (x+11, y+4)$
11. $(x,y) \rightarrow (x, y-3)$
12. $(x,y) \rightarrow (x-5, y+8)$

In questions 13-16, $\triangle A'B'C'$ is the image of $\triangle ABC$. Write the translation rule.

13. 

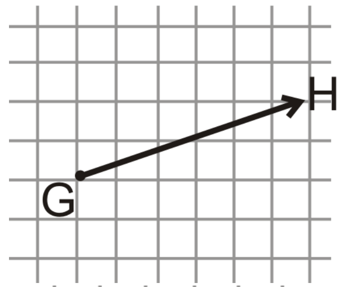
14. 

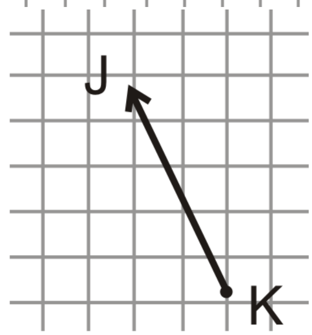
15. 

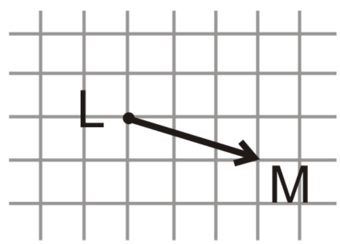
16. 

17. Verify that a translation is an isometry using the triangle from #15.
18. If $\triangle A'B'C'$ was the *preimage* and $\triangle ABC$ was the image, write the translation rule for #16.

For questions 19-21, name each vector and find its component form.

19. 

20. 

21. 

For questions 22-24, plot and correctly label each vector.

22. $\overrightarrow{AB} = \langle 4, -3 \rangle$
23. $\overrightarrow{CD} = \langle -6, 8 \rangle$
24. $\overrightarrow{FE} = \langle -2, 0 \rangle$
25. The coordinates of $\triangle DEF$ are $D(4,-2), E(7,-4)$ and $F(5,3)$. Translate $\triangle DEF$ using the vector $\langle 5, 11 \rangle$ and find the coordinates of $\triangle D'E'F'$.
26. The coordinates of quadrilateral $QUAD$ are $Q(-6,1), U(-3,7), A(4,-2)$ and $D(1,-8)$. Translate $QUAD$ using the vector $\langle -3, -7 \rangle$ and find the coordinates of $Q'U'A'D'$.

For problems 27-29, write the translation rule as a translation vector.

27. $(x,y) \rightarrow (x-3, y+8)$
28. $(x,y) \rightarrow (x+9, y-12)$
29. $(x,y) \rightarrow (x, y-7)$

For problems 30-32, write the translation vector as a translation rule.

30. $\langle -7, 2 \rangle$
31. $\langle 11, 25 \rangle$
32. $\langle 15, -9 \rangle$

Review Queue Answers

a. $y = -2x + 17$
b. Kite
c. $y = -2x + 5$
d. $y = \frac{1}{2}x - 5$

9.3 Reflections

Learning Objectives

- Reflect a figure over a given line.
- Determine the rules of reflections in the coordinate plane.

Review Queue

a. Define reflection in your own words.
b. Plot $A(-3,2)$. Translate A such that $(x,y) \rightarrow (x+6,y)$.
c. What line is halfway between A and A' ?
d. Translate A such that $(x,y) \rightarrow (x,y-4)$. Call this point A''.
e. What line is halfway between A and A'' ?

Know What? A lake can act like a mirror in nature. Describe the line of reflection in the photo below. If this image were on the coordinate plane, what could the equation of the line of reflection be? (There could be more than one correct answer, depending on where you place the origin.)

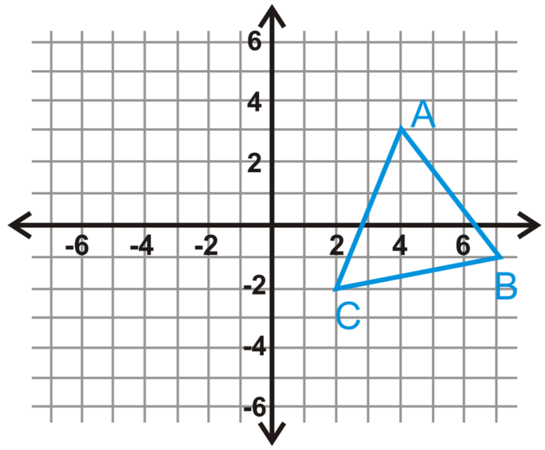
Reflections over an Axis

The next transformation is a reflection. Another way to describe a reflection is a "flip."

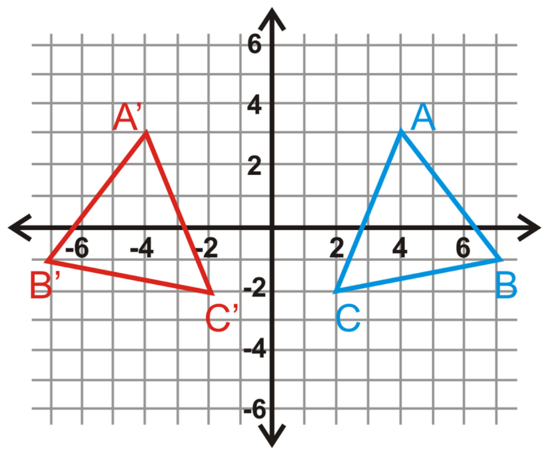
Reflection: A transformation that turns a figure into its mirror image by flipping it over a line.

Line of Reflection: The line that a figure is reflected over.

Example 1: Reflect $\triangle ABC$ over the $y-$axis. Find the coordinates of the image.



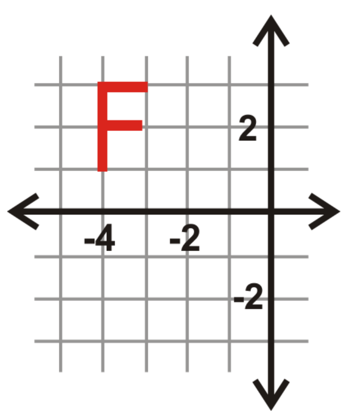
Solution: To reflect $\triangle ABC$ over the $y-$axis the $y-$coordinates will remain the same. The $x-$coordinates will be the same distance away from the $y-$axis, but on the other side of the $y-$axis.



$$A(4,3) \rightarrow A'(-4,3)$$
$$B(7,-1) \rightarrow B'(-7,-1)$$
$$C(2,-2) \rightarrow C'(-2,-2)$$

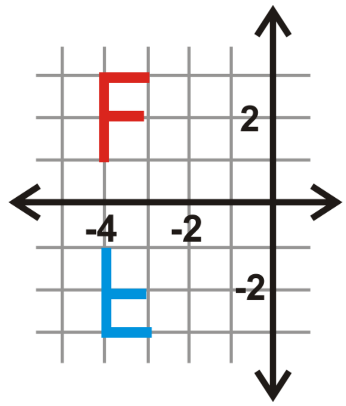
From this example, we can generalize a rule for reflecting a figure over the $y-$axis.

Reflection over the $y-$**axis:** If (x,y) is reflected over the $y-$axis, then the image is $(-x,y)$.



Example 2: Reflect the letter "F" over the $x-$axis.

Solution: To reflect the letter F over the $x-$axis, now the $x-$coordinates will remain the same and the $y-$coordinates will be the same distance away from the $x-$axis on the other side.



The generalized rule for reflecting a figure over the $x-$axis:

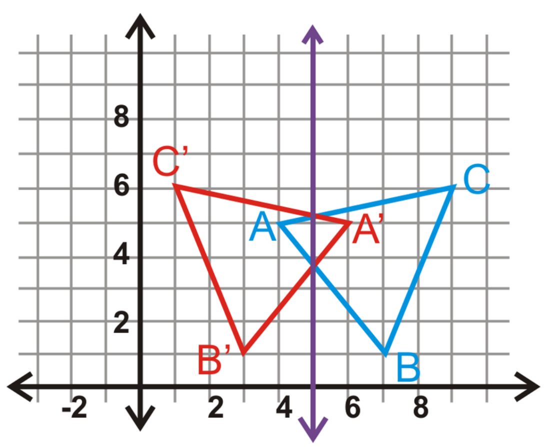
Reflection over the $x-$axis: If (x,y) is reflected over the $x-$axis, then the image is $(x,-y)$.

Reflections over Horizontal and Vertical Lines

Other than the x and y axes, we can reflect a figure over any vertical or horizontal line.

Example 3: Reflect the triangle $\triangle ABC$ with vertices $A(4,5), B(7,1)$ and $C(9,6)$ over the line $x = 5$.

Solution: Notice that this vertical line is through our preimage. Therefore, the image's vertices are the same distance away from $x = 5$ as the preimage. As with reflecting over the $y-$axis (or $x = 0$), the $y-$coordinates will stay the same.

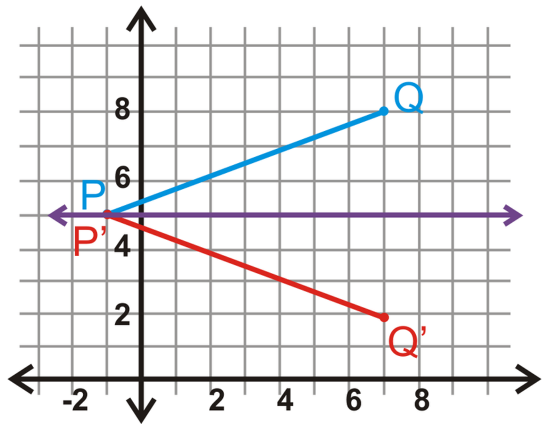


$$A(4,5) \rightarrow A'(6,5)$$
$$B(7,1) \rightarrow B'(3,1)$$
$$C(9,6) \rightarrow C'(1,6)$$

Example 4: Reflect the line segment $\overline{PQ}$ with endpoints $P(-1,5)$ and $Q(7,8)$ over the line $y = 5$.

Solution: Here, the line of reflection is on P, which means P' has the same coordinates. Q' has the same $x-$coordinate as Q and is the same distance away from $y = 5$, but on the other side.

$$P(-1,5) \rightarrow P'(-1,5)$$
$$Q(7,8) \rightarrow Q'(7,2)$$

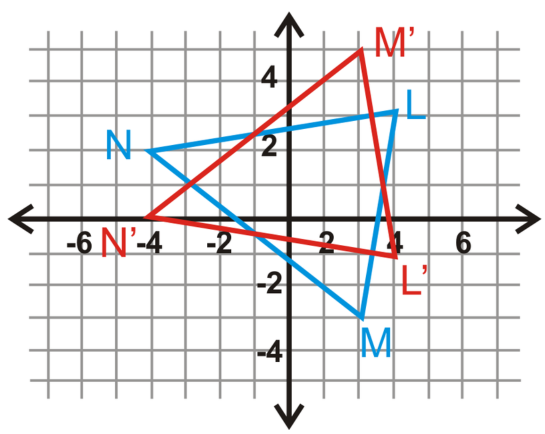


Reflection over $x = a$: If (x,y) is reflected over the vertical line $x = a$, then the image is $(2a-x,y)$.

Reflection over $y = b$: If (x,y) is reflected over the horizontal line $y = b$, then the image is $(x,2b-y)$.

From these examples we also learned that if a point is on the line of reflection then the image is the same as the original point.

Example 5: A triangle $\triangle LMN$ and its reflection, $\triangle L'M'N'$ are to the left. What is the line of reflection?



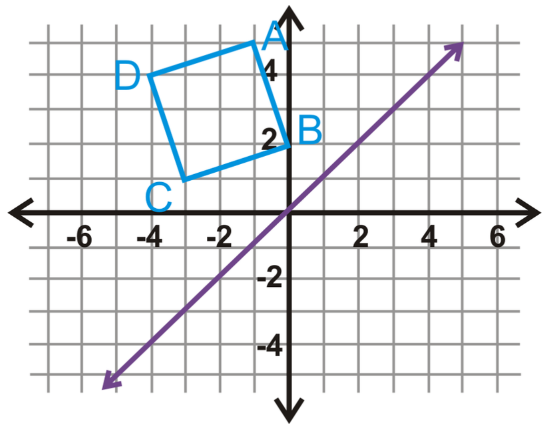
Solution: Looking at the graph, we see that the preimage and image intersect when $y = 1$. Therefore, this is the line of reflection.

If the image does not intersect the preimage, find the midpoint between a preimage and its image. This point is on the line of reflection. You will need to determine if the line is vertical or horizontal.

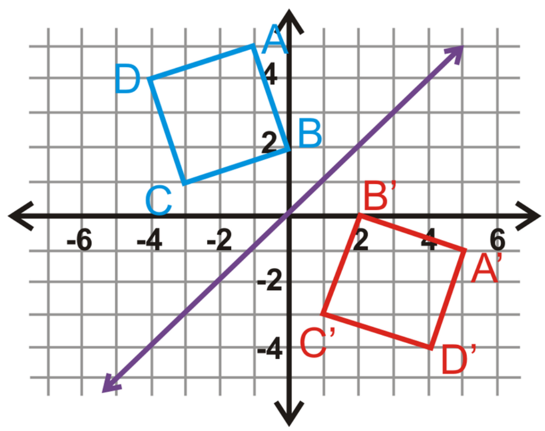
Reflections over $y = x$ **and** $y = -x$

Technically, any line can be a line of reflection. We are going to study two more cases of reflections, reflecting over $y = x$ and over $y = -x$.

Example 6: Reflect square $ABCD$ over the line $y = x$.



Solution: The purple line is $y = x$. To reflect an image over a line that is not vertical or horizontal, you can fold the graph on the line of reflection.

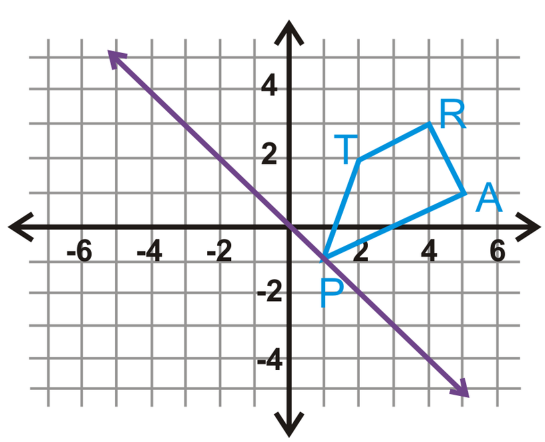


$$A(-1,5) \rightarrow A'(5,-1)$$
$$B(0,2) \rightarrow B'(2,0)$$
$$C(-3,1) \rightarrow C'(1,-3)$$
$$D(-4,4) \rightarrow D'(4,-4)$$

From this example, we see that the x and y values are switched when a figure is reflected over the line $y = x$.

Reflection over $y = x$: If (x,y) is reflected over the line $y = x$, then the image is (y,x).

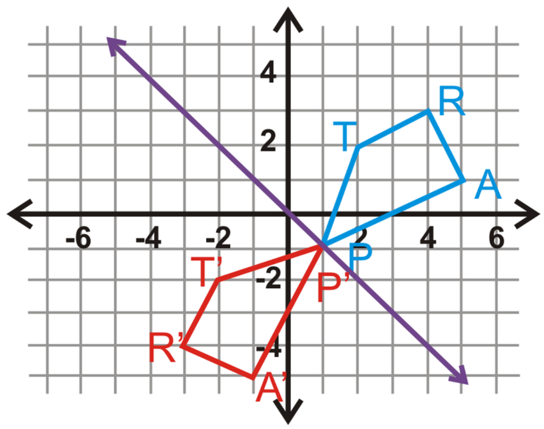
Example 7: Reflect the trapezoid *TRAP* over the line $y = -x$.



Solution: The purple line is $y=-x$. You can reflect the trapezoid over this line just like we did in Example 6.

$$\begin{aligned}T(2,2)&\rightarrow T'(-2,-2)\\R(4,3)&\rightarrow R'(-3,-4)\\A(5,1)&\rightarrow A'(-1,-5)\\P(1,-1)&\rightarrow P'(1,-1)\end{aligned}$$

From this example, we see that the x and y values are switched and the signs are changed when a figure is reflected over the line $y=x$.



Reflection over $y=-x$: If (x,y) is reflected over the line $y=-x$, then the image is $(-y,-x)$.

At first glance, it does not look like P and P' follow the rule above. However, when you switch 1 and -1 you would have (-1, 1). Then, take the opposite sign of both, (1, -1). Therefore, when a point is on the line of reflection, it will be its own reflection.

From all of these examples, we notice that ***a reflection is an isometry***.

Know What? Revisited The white line in the picture is the line of reflection. This line coincides with the water's edge. If we were to place this picture on the coordinate plane, the line of reflection would be any horizontal line. One example could be the $x-$axis.

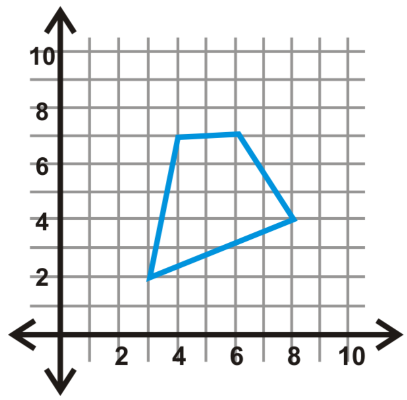
Review Questions

1. Which letter is a reflection over a vertical line of the letter "b"?

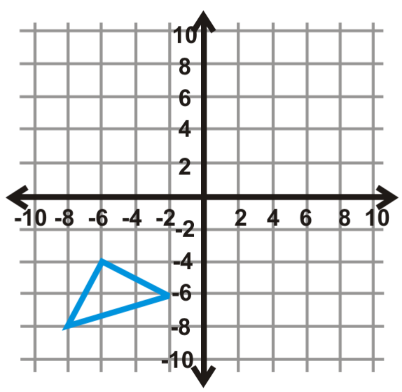
2. Which letter is a reflection over a horizontal line of the letter "*b*"?

Reflect each shape over the given line.

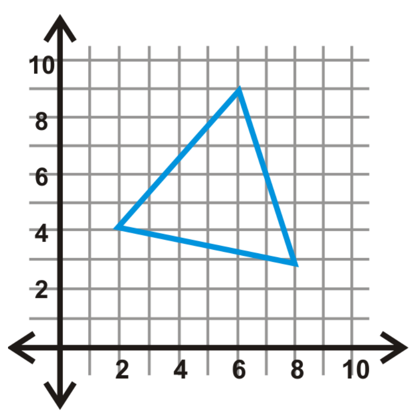
3. $y-$axis



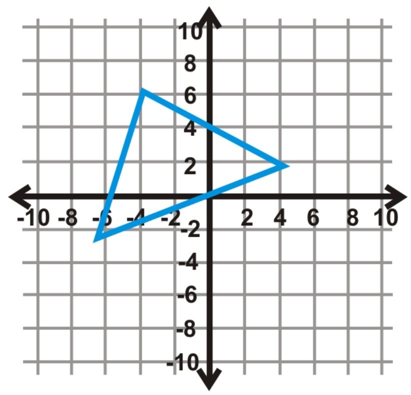
4. $x-$axis



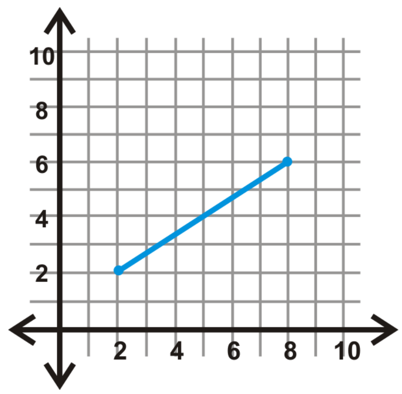
5. $y=3$



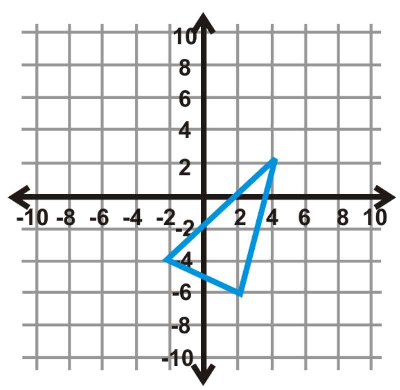
6. $x=-1$



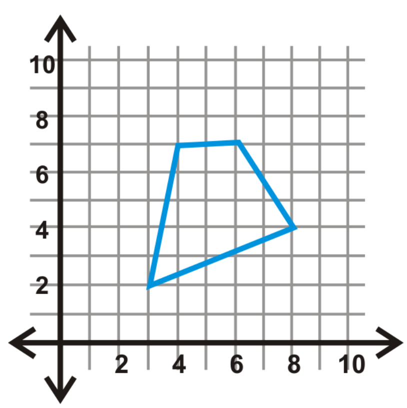
7. $x-$axis



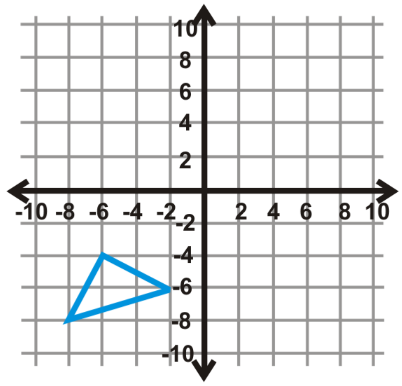
8. $y-$axis



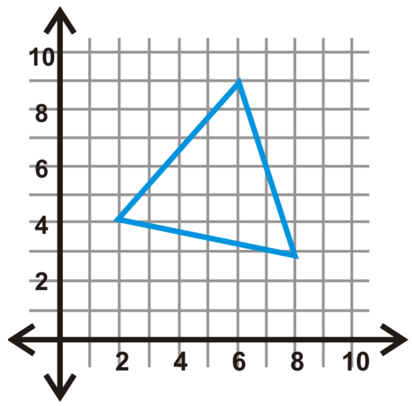
9. $y = x$



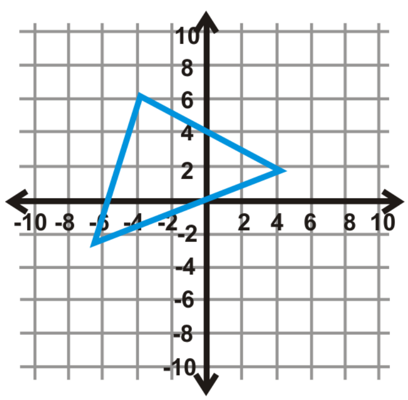
10. $y = -x$



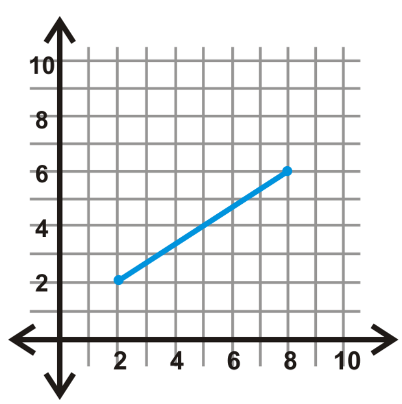
11. $x = 2$



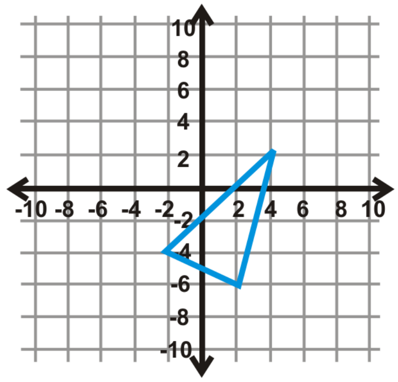
12. $y = -4$



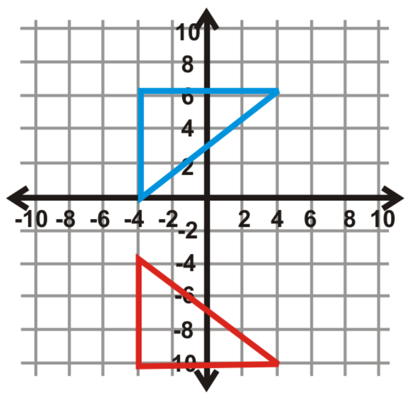
13. $y = -x$



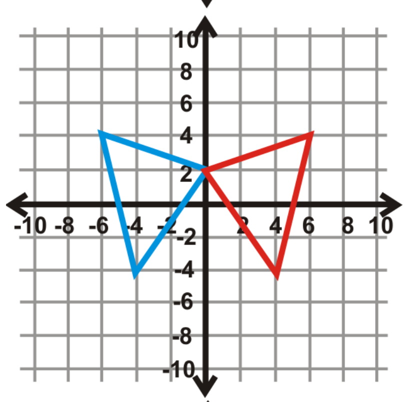
14. $y = x$



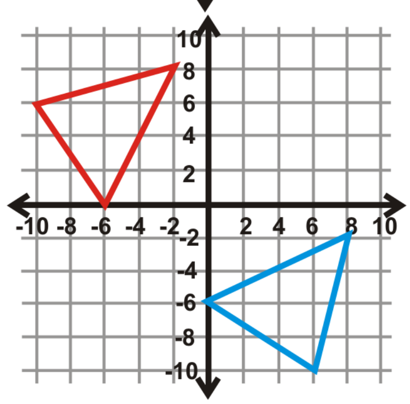
Find the line of reflection of the blue triangle (preimage) and the red triangle (image).



15.



16.



17.

Two Reflections The vertices of $\triangle ABC$ are $A(-5,1), B(-3,6)$, and $C(2,3)$. Use this information to answer questions 18-21.

18. Plot $\triangle ABC$ on the coordinate plane.
19. Reflect $\triangle ABC$ over $y = 1$. Find the coordinates of $\triangle A'B'C'$.
20. Reflect $\triangle A'B'C'$ over $y = -3$. Find the coordinates of $\triangle A''B''C''$.
21. What ***one*** transformation would be the same as this double reflection?

Two Reflections The vertices of $\triangle DEF$ are $D(6,-2), E(8,-4)$, and $F(3,-7)$. Use this information to answer questions 22-25.

22. Plot $\triangle DEF$ on the coordinate plane.
23. Reflect $\triangle DEF$ over $x = 2$. Find the coordinates of $\triangle D'E'F'$.
24. Reflect $\triangle D'E'F'$ over $x = -4$. Find the coordinates of $\triangle D''E''F''$.
25. What ***one*** transformation would be the same as this double reflection?

Two Reflections The vertices of $\triangle GHI$ are $G(1,1), H(5,1)$, and $I(5,4)$. Use this information to answer questions 26-29.

26. Plot $\triangle GHI$ on the coordinate plane.
27. Reflect $\triangle GHI$ over the $x-$axis. Find the coordinates of $\triangle G'H'I'$.
28. Reflect $\triangle G'H'I'$ over the $y-$axis. Find the coordinates of $\triangle G''H''I''$.
29. What ***one*** transformation would be the same as this double reflection?
30. Following the steps to reflect a triangle using a compass and straightedge.
    a. Make a triangle on a piece of paper. Label the vertices A, B and C.
    b. Make a line next to your triangle (this will be your line of reflection).
    c. Construct perpendiculars from each vertex of your triangle through the line of reflection.
    d. Use your compass to mark off points on the other side of the line that are the same distance from the line as the original A, B and C. Label the points A', B' and C'.
    e. Connect the new points to make the image $\triangle A'B'C'$.
31. Describe the relationship between the line of reflection and the segments connecting the preimage and image points.
32. Repeat the steps from problem 28 with a line of reflection that passes ***through*** the triangle.

Review Queue Answers

a. *Examples are:* To flip an image over a line; A mirror image.
b. $A'(3,2)$
c. the $y-$axis
d. $A''(-3,-2)$
e. the $x-$axis

9.4 Rotations

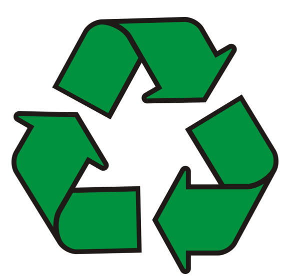
Learning Objectives

- Find the image of a figure in a rotation in a coordinate plane.
- Recognize that a rotation is an isometry.

Review Queue

a. Reflect $\triangle XYZ$ with vertices $X(9,2), Y(2,4)$ and $Z(7,8)$ over the $y-$axis. What are the vertices of $\triangle X'Y'Z'$?
b. Reflect $\triangle X'Y'Z'$ over the $x-$axis. What are the vertices of $\triangle X''Y''Z''$?
c. How do the coordinates of $\triangle X''Y''Z''$ relate to $\triangle XYZ$?

Know What? The international symbol for recycling appears below. It is three arrows rotated around a point. Let's assume that the arrow on the top is the preimage and the other two are its images. Find the center of rotation and the angle of rotation for each image.

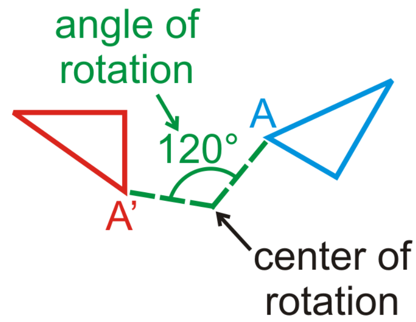


Defining Rotations

Rotation: A transformation by which a figure is turned around a fixed point to create an image.

Center of Rotation: The fixed point that a figure is rotated around.

Lines can be drawn from the preimage to the center of rotation, and from the center of rotation to the image. The angle formed by these lines is the angle of ***rotation.***

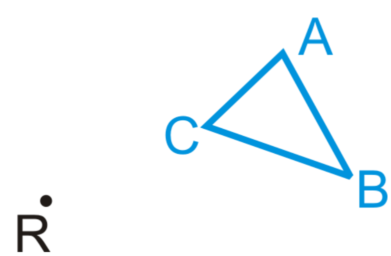


In this section, our center of rotation will always be the ***origin.*** Rotations can also be clockwise or counterclockwise. We will only do ***counterclockwise*** rotations, to go along with the way the quadrants are numbered.

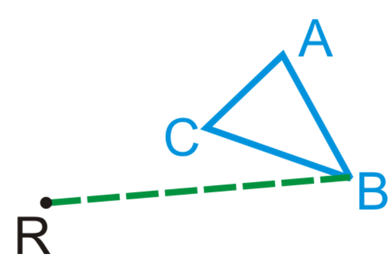
Investigation 12-1: Drawing a Rotation of 100°

Tools Needed: pencil, paper, protractor, ruler

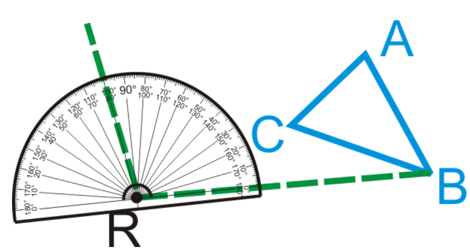
a. Draw $\triangle ABC$ and a point R outside the circle.



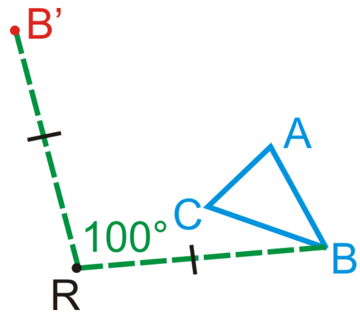
b. Draw the line segment $\overline{RB}$.

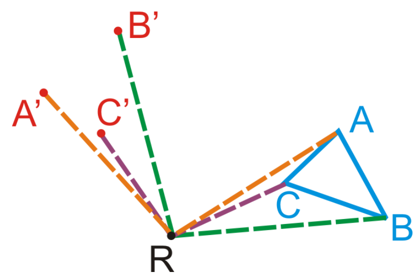


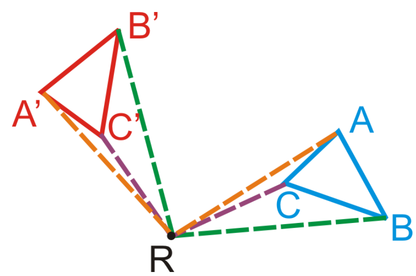
c. Take your protractor, place the center on R and the initial side on $\overline{RB}$. Mark a 100° angle.



d. Find B' such that $RB = RB'$.
e. Repeat steps 2-4 with points A and C.
f. Connect A', B', and C' to form $\triangle A'B'C'$.





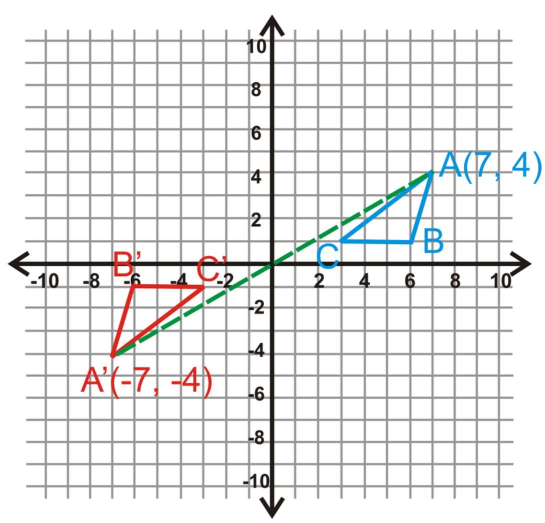


This is the process you would follow to rotate any figure 100° counterclockwise. If it was a different angle measure, then in Step 3, you would mark a different angle. You will need to repeat steps 2-4 for every vertex of the shape.

180° **Rotation**

To rotate a figure 180° in the coordinate plane, we use the origin as the center of the rotation. Recall, that a 180° angle is the same as a straight line. So, a rotation of a point over the origin of 180° will be on the same line and the same distance away from the origin.

Example 1: Rotate $\triangle ABC$, with vertices $A(7,4), B(6,1)$, and $C(3,1)$ 180°. Find the coordinates of $\triangle A'B'C'$.



Solution: You can either use Investigation 12-1 or the hint given above to find $\triangle A'B'C'$. It is very helpful to graph the triangle. Using the hint, if A is $(7,4)$, that means it is 7 units to the right of the origin and 4 units up. A' would then be 7 units to the ***left*** of the origin and 4 units ***down.*** The vertices are:

$$A(7,4) \rightarrow A'(-7,-4)$$
$$B(6,1) \rightarrow B'(-6,-1)$$
$$C(3,1) \rightarrow C'(-3,-1)$$

The image has vertices that are the negative of the preimage. This will happen every time a figure is rotated 180°.

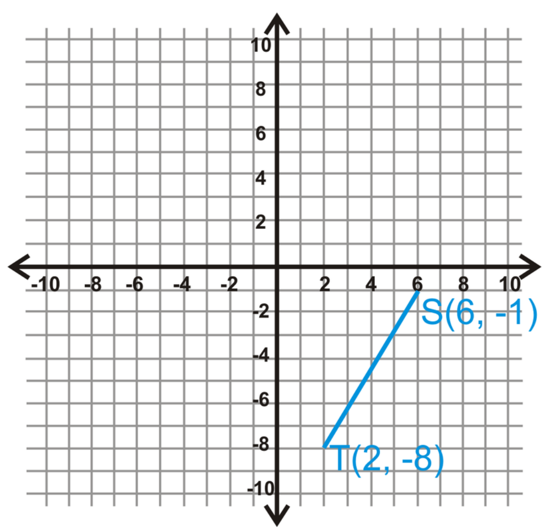
Rotation of 180°**:** If (x,y) is rotated 180° around the origin, then the image will be $(-x,-y)$.

From this example, we can also see that ***a rotation is an isometry.*** This means that $\triangle ABC \cong \triangle A'B'C'$. You can use the distance formula to verify that our assertion holds true.

90° **Rotation**

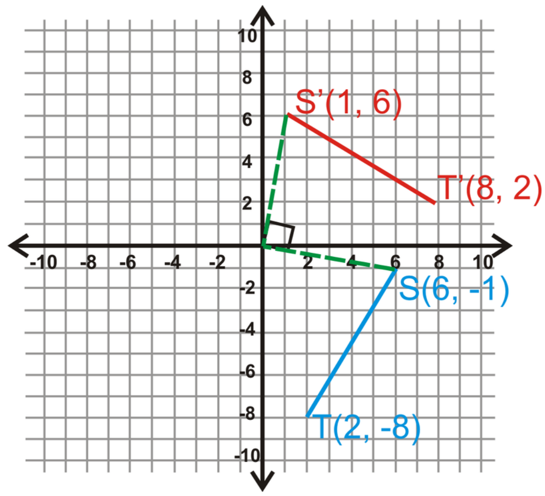
Similar to the 180° rotation, a 90° rotation (counterclockwise) is an isometry. Each image will be the same distance away from the origin as its preimage, but rotated 90°.

Example 2: Rotate $\overline{ST}$ 90°.



Solution: When we rotate something 90°, you can use Investigation 12-1. Draw lines from the origin to S and T. The line from each point to the origin is going to be ***perpendicular*** to the line from the origin to its image. Therefore, if S is 6 units to the ***right*** of the origin and 1 unit ***down,*** S' will be 6 units ***up*** and 1 to the ***right.***

Using this pattern, T' is (8, 2).



If you were to write the slope of each point to the origin, S would be $\frac{-1}{6} \rightarrow \frac{y}{x}$, and S' must be $\frac{6}{1} \rightarrow \frac{y'}{x'}$. Again, they are perpendicular slopes, following along with the 90° rotation. Therefore, the x and the y values switch and the new $x-$value is the opposite sign of the original $y-$value.

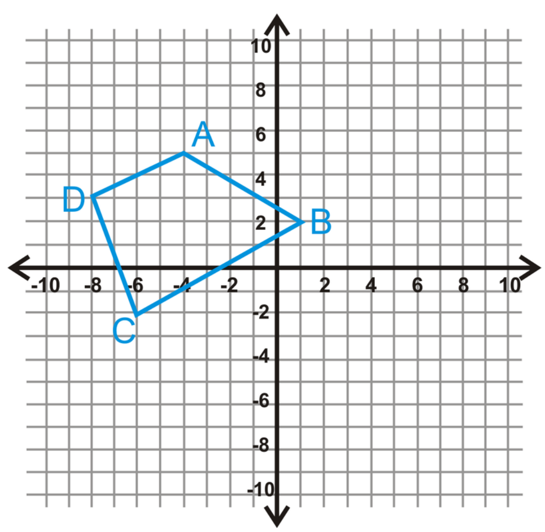
Rotation of 90°**:** If (x, y) is rotated 90° around the origin, then the image will be $(-y, x)$.

Rotation of 270°

A rotation of 270° counterclockwise would be the same as a clockwise rotation of 90°. We also know that a 90° rotation and a 270° rotation are 180° apart. We know that for every 180° rotation, the x and y values are negated. So, if the values of a 90° rotation are $(-y, x)$, then a 270° rotation would be the opposite sign of each, or $(y, -x)$.

Rotation of 270°**:** If (x, y) is rotated 270° around the origin, then the image will be $(y, -x)$.

Example 3: Find the coordinates of $ABCD$ after a 270° rotation.



Solution: Using the rule, we have:

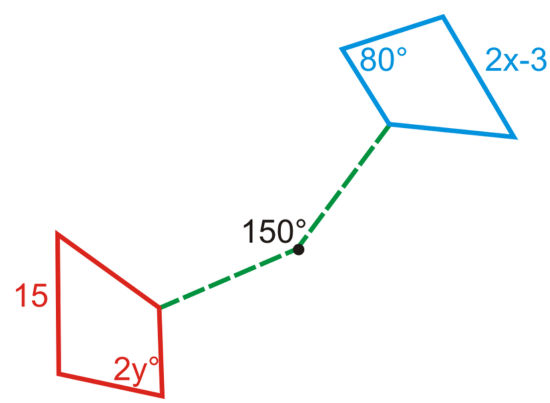
$$\begin{aligned}(x,y) &\rightarrow (y,-x)\\ A(-4,5) &\rightarrow A'(5,4)\\ B(1,2) &\rightarrow B'(2,-1)\\ C(-6,-2) &\rightarrow C'(-2,6)\\ D(-8,3) &\rightarrow D'(3,8)\end{aligned}$$

While we can rotate any image any amount of degrees, only $90^\circ, 180^\circ$ and 270° have special rules. To rotate a figure by an angle measure other than these three, you must use Investigation 12-1.

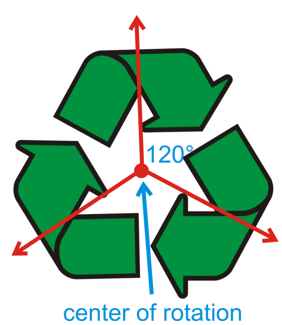
Example 4: ***Algebra Connection*** The rotation of a quadrilateral is shown below. What is the measure of x and y?

Solution: Because a rotation is an isometry, we can set up two equations to solve for x and y.

$$\begin{aligned}2y &= 80^\circ\\ y &= 40^\circ\end{aligned} \qquad \begin{aligned}2x-3 &= 15\\ 2x &= 18\\ x &= 9\end{aligned}$$



Know What? Revisited The center of rotation is shown in the picture below. If we draw rays to the same point in each arrow, we see that the two images are a 120° rotation in either direction.

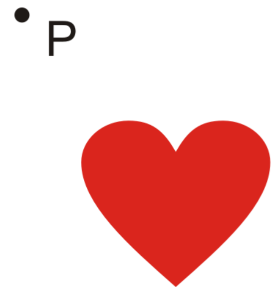


Review Questions

In the questions below, every rotation is *counterclockwise,* unless otherwise stated.

Using Investigation 12-1, rotate each figure around point P the given angle measure.

1. 50°



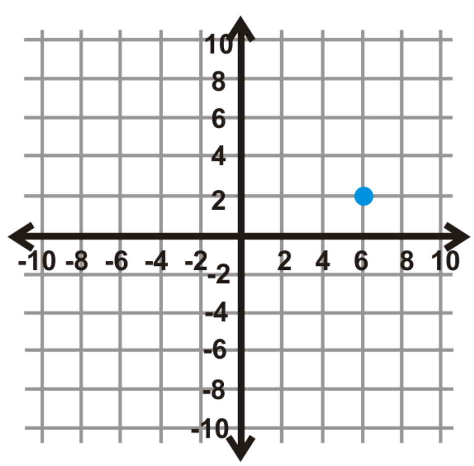
2. 120°

3. 200°

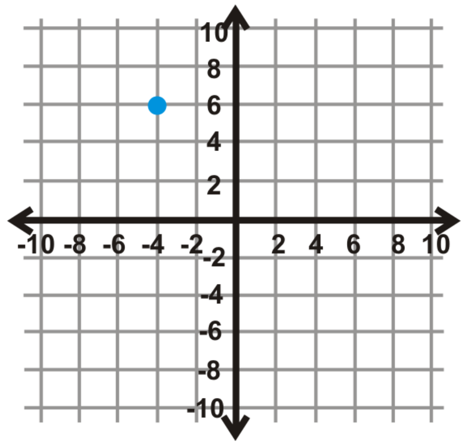
4. If you rotated the letter *p* 180° counterclockwise, what letter would you have?
5. If you rotated the letter *p* 180° *clockwise,* what letter would you have? Why do you think that is?
6. A 90° clockwise rotation is the same as what counterclockwise rotation?
7. A 270° clockwise rotation is the same as what counterclockwise rotation?
8. Rotating a figure 360° is the same as what other rotation?

Rotate each figure in the coordinate plane the given angle measure. The center of rotation is the origin.

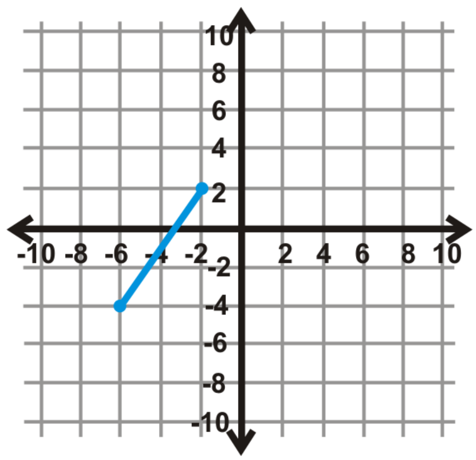
9. 180°



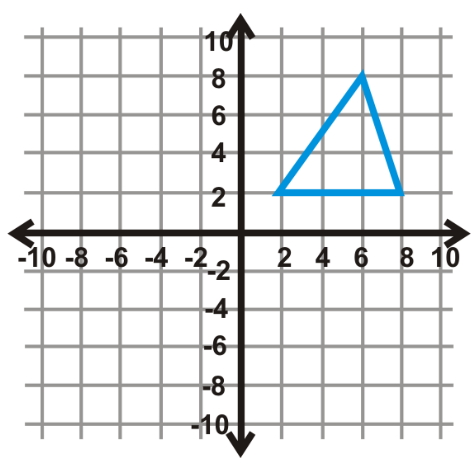
10. 90°



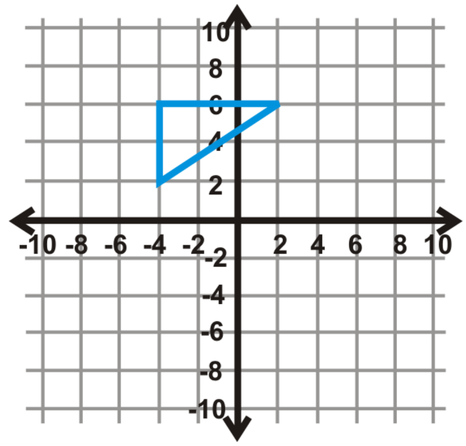
11. 180°



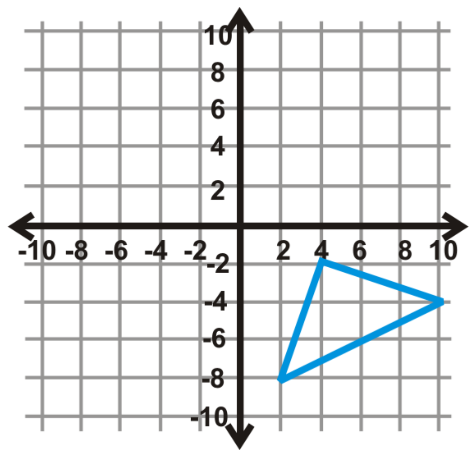
12. 270°



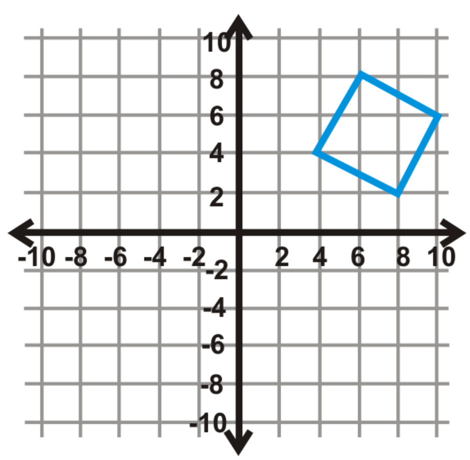
13. 90°



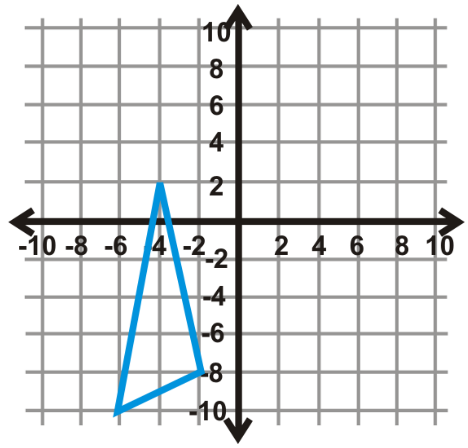
14. 270°



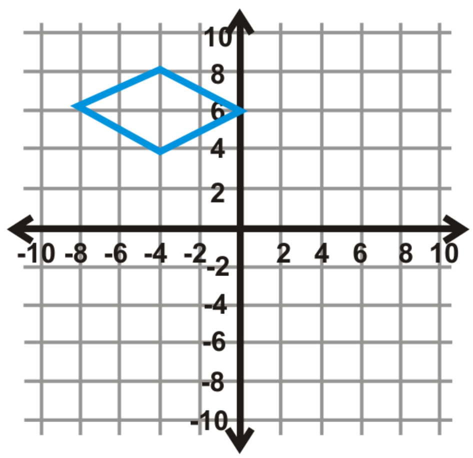
15. 180°



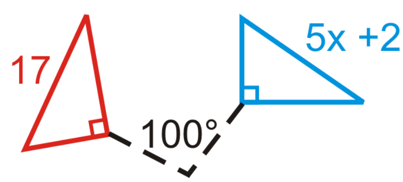
16. 270°

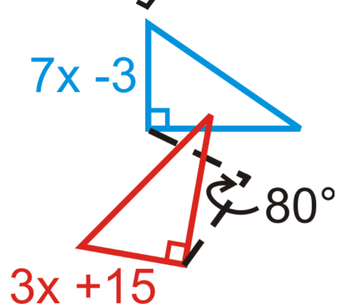


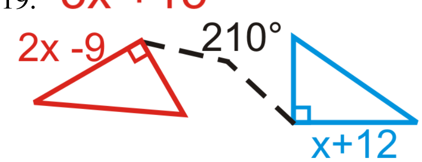
17. 90°



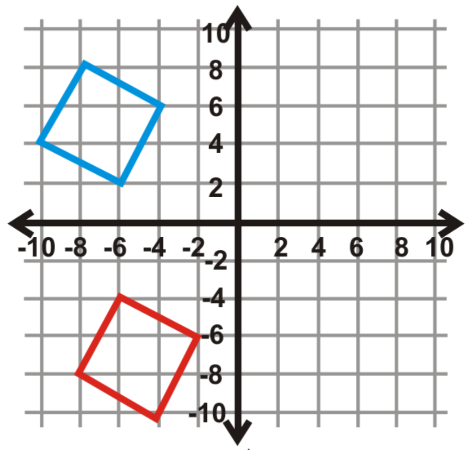
Algebra Connection Find the measure of *x* in the rotations below. The blue figure is the preimage.

18. 

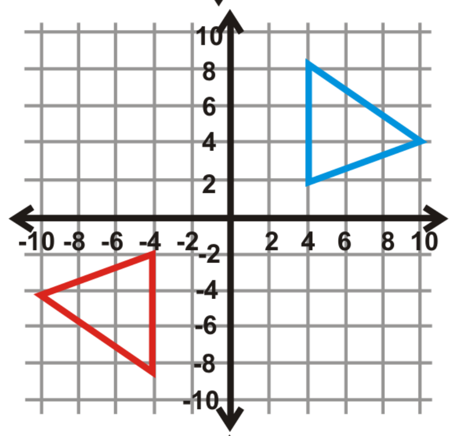
19. 

20. 

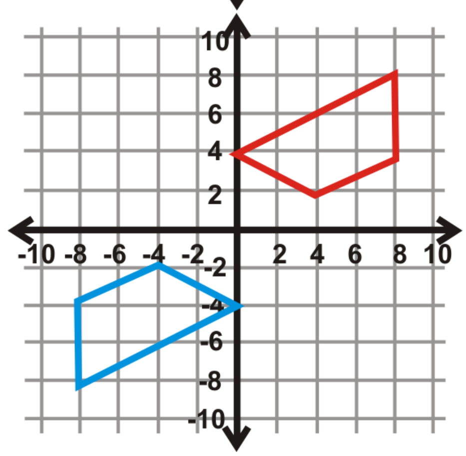
Find the angle of rotation for the graphs below. The center of rotation is the origin and the blue figure is the preimage.



21.



22.



23.

Two Reflections The vertices of $\triangle GHI$ are $G(-2,2), H(8,2)$ and $I(6,8)$. Use this information to answer questions 24-27.

24. Plot $\triangle GHI$ on the coordinate plane.
25. Reflect $\triangle GHI$ over the $x-$axis. Find the coordinates of $\triangle G'H'I'$.
26. Reflect $\triangle G'H'I'$ over the $y-$axis. Find the coordinates of $\triangle G''H''I''$.
27. What ***one*** transformation would be the same as this double reflection?

Multistep Construction Problem

28. Draw two lines that intersect, m and n, and $\triangle ABC$. Reflect $\triangle ABC$ over line m to make $\triangle A'B'C'$. Reflect $\triangle A'B'C'$ over line n to get $\triangle A''B''C''$. Make sure $\triangle ABC$ does not intersect either line.
29. Draw segments from the intersection point of lines m and n to A and A''. Measure the angle between these segments. This is the angle of rotation between $\triangle ABC$ and $\triangle A''B''C''$.

30. Measure the angle between lines m and n. Make sure it is the angle which contains $\triangle A'B'C'$ in the interior of the angle.
31. What is the relationship between the angle of rotation and the angle between the two lines of reflection?

Review Queue Answers

a. $X'(-9,2), Y'(-2,4), Z'(-7,8)$
b. $X''(-9,-2), Y''(-2,-4), Z''(-7,-8)$
c. $\triangle X''Y''Z''$ is the double negative of $\triangle XYZ; (x,y) \rightarrow (-x,-y)$

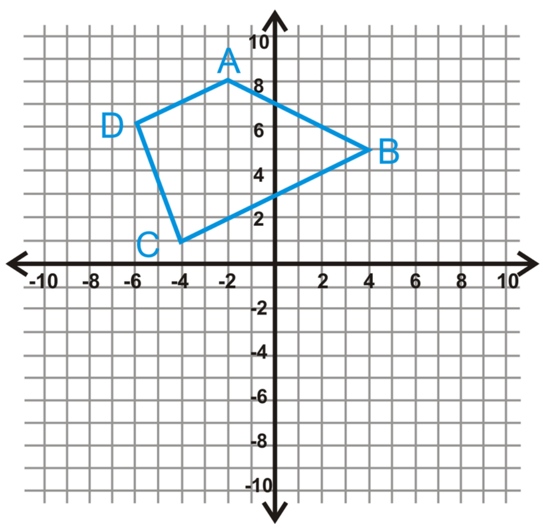
9.5 Composition of Transformations

Learning Objectives

- Perform a glide reflection.
- Perform a reflection over parallel lines and the axes.
- Perform a double rotation with the same center of rotation.
- Determine a single transformation that is equivalent to a composite of two transformations.

Review Queue

a. Reflect $ABCD$ over the $x-$axis. Find the coordinates of $A'B'C'D'$.

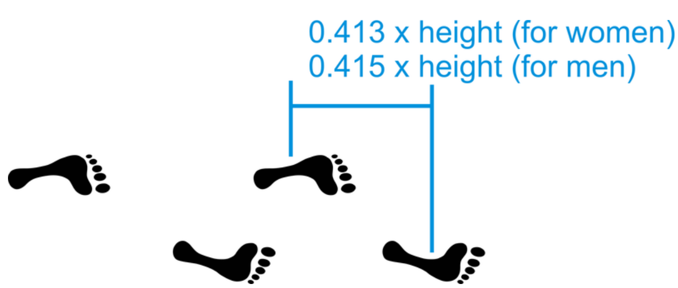


b. Translate $A'B'C'D'$ such that $(x,y) \rightarrow (x+4,y)$. Find the coordinates of $A''B''C''D''$.

c. Now, start over. Translate $ABCD$ such that $(x,y) \rightarrow (x+4,y)$. Find the coordinates of $A'B'C'D'$.

d. Reflect $A'B'C'D'$ from #3 over the $x-$axis. Find the coordinates of $A''B''C''D''$. Are they the same as #2?

Know What? An example of a glide reflection is your own footprint. The equations to find your average footprint are in the diagram below. Determine your average footprint and write the rule for one stride. You may assume your stride starts at (0, 0).



Glide Reflections

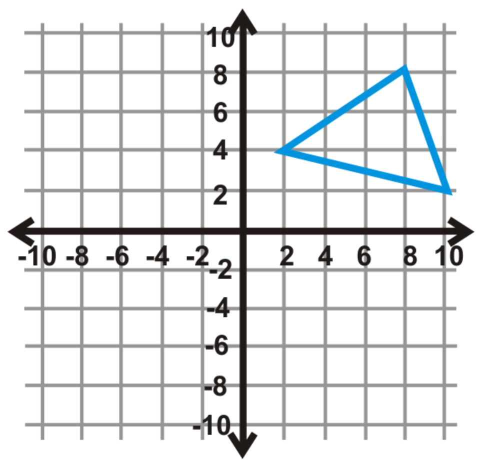
Now that we have learned all our rigid transformations, or isometries, we can perform more than one on the same figure. In your homework last night you actually performed a composition of two reflections. And, in the Review Queue above, you performed a composition of a reflection and a translation.

Composition (of transformations): To perform more than one rigid transformation on a figure.

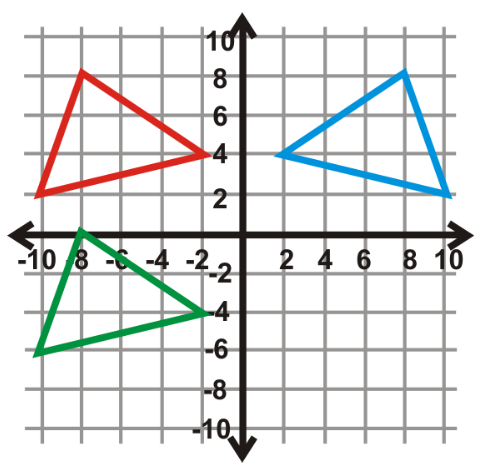
Glide Reflection: A composition of a reflection and a translation. The translation is in a direction parallel to the line of reflection.

So, in the Review Queue above, you performed a glide reflection on $ABCD$. Hopefully, in #4, you noticed that ***the order in which you reflect or translate does not matter.*** It is important to note that the translation for any glide reflection will always be in one direction. So, if you reflect over a vertical line, the translation can be up or down, and if you reflect over a horizontal line, the translation will be to the left or right.

Example 1: Reflect $\triangle ABC$ over the $y-$axis and then translate the image 8 units down.



Solution: The green image below is the final answer.



$$A(8,8) \rightarrow A''(-8,0)$$
$$B(2,4) \rightarrow B''(-2,-4)$$
$$C(10,2) \rightarrow C''(-10,-6)$$

One of the interesting things about compositions is that they can always be written as one rule. What this means is, you don't necessarily have to perform one transformation followed by the next. You can write a rule and perform them at the same time.

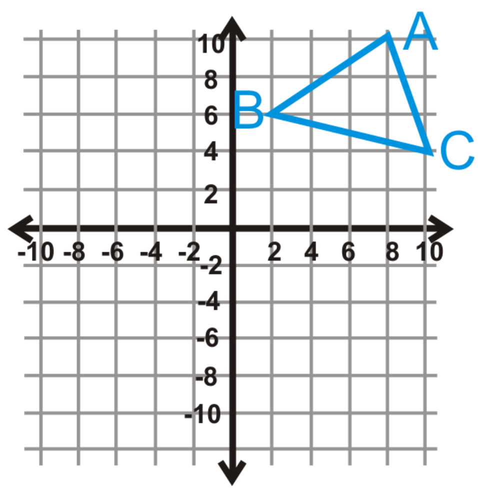
Example 2: Write a single rule for $\triangle ABC$ to $\triangle A''B''C''$ from Example 1.

Solution: Looking at the coordinates of A to A'', the $x-$value is the opposite sign and the $y-$value is $y-8$. Therefore the rule would be $(x,y) \rightarrow (-x,y-8)$.

Notice that this follows the rules we have learned in previous sections about a reflection over the $y-$axis and translations.

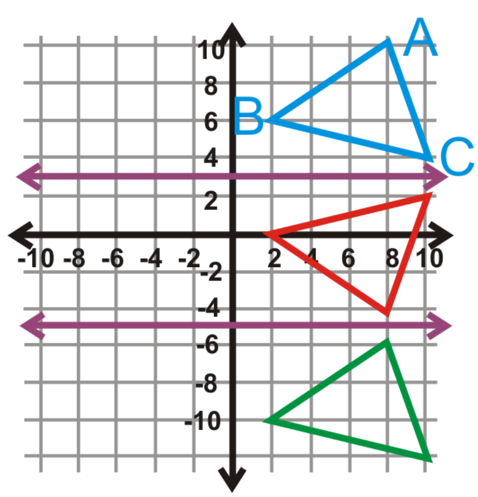
Reflections over Parallel Lines

The next composition we will discuss is a double reflection over parallel lines. For this composition, we will only use horizontal or vertical lines.



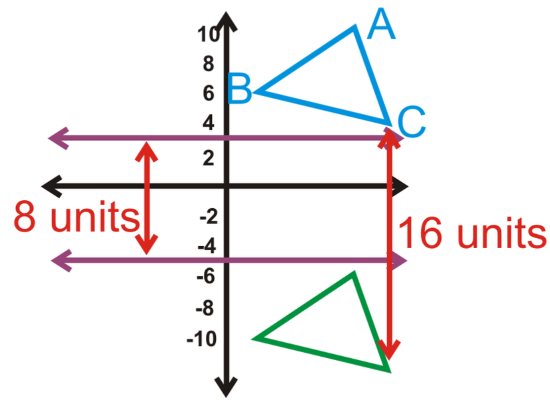
Example 3: Reflect $\triangle ABC$ over $y = 3$ and $y = -5$.

Solution: Unlike a glide reflection, order matters. Therefore, you would reflect over $y = 3$ first, followed by a reflection of this image (red triangle) over $y = -5$. Your answer would be the green triangle in the graph below.



Example 4: Write a single rule for $\triangle ABC$ to $\triangle A''B''C''$ from Example 3.

Solution: Looking at the graph below, we see that the two lines are 8 units apart and the figures are 16 units apart. Therefore, the double reflection is the same as a single translation that is double the distance between the two lines.



$$(x,y) \rightarrow (x,y-16)$$

Reflections over Parallel Lines Theorem: If you compose two reflections over parallel lines that are h units apart, it is the same as a single translation of $2h$ units.

Be careful with this theorem. Notice, it does not say which direction the translation is in. So, to apply this theorem, you would still need to visualize, or even do, the reflections to see in which direction the translation would be.

Example 5: $\triangle DEF$ has vertices $D(3,-1), E(8,-3)$, and $F(6,4)$. Reflect $\triangle DEF$ over $x=-5$ and $x=1$. This double reflection would be the same as which one translation?

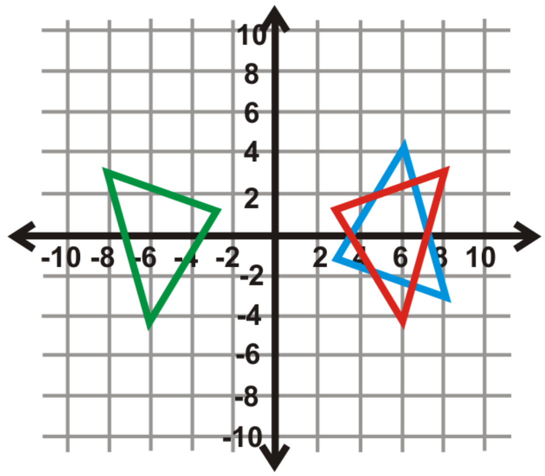
Solution: From the Reflections over Parallel Lines Theorem, we know that this double reflection is going to be the same as a single translation of $2(1-(-5))$ or 12 units. Now, we need to determine if it is to the right or to the left. Because we first reflect over a line that is further away from $\triangle DEF$, to the ***left***, $\triangle D''E''F''$ will be on the ***right*** of $\triangle DEF$. So, it would be the same as a translation of 12 units to the right. If the lines of reflection were switched and we reflected the triangle over $x=1$ followed by $x=-5$, then it would have been the same as a translation of 12 units to the ***left.***

Reflections over the x and y Axes

You can also reflect over intersecting lines. First, we will reflect over the x and y axes.

Example 6: Reflect $\triangle DEF$ from Example 5 over the $x-$axis, followed by the $y-$axis. Determine the coordinates of $\triangle D''E''F''$ and what one transformation this double reflection would be the same as.

Solution: $\triangle D''E''F''$ is the green triangle in the graph below. If we compare the coordinates of it to $\triangle DEF$, we have:



$$D(3,-1) \to D'(-3,1)$$
$$E(8,-3) \to E'(-8,3)$$
$$F(6,4) \to F'(-6,-4)$$

If you recall the rules of rotations from the previous section, this is the same as a rotation of 180°.

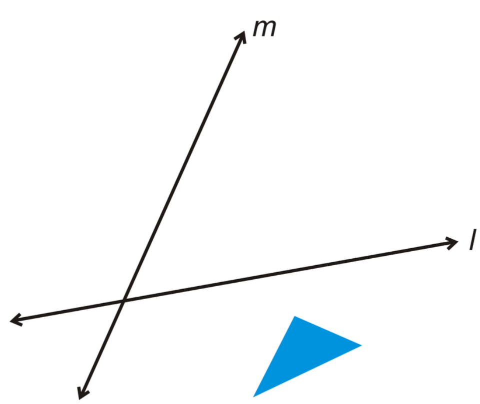
Reflection over the Axes Theorem: If you compose two reflections over each axis, then the final image is a rotation of 180° of the original.

With this particular composition, order does not matter. Let's look at the angle of intersection for these lines. We know that the axes are perpendicular, which means they intersect at a 90° angle. The final answer was a rotation of 180°, which is double 90°. Therefore, we could say that the composition of the reflections over each axis is a rotation of double their angle of intersection.

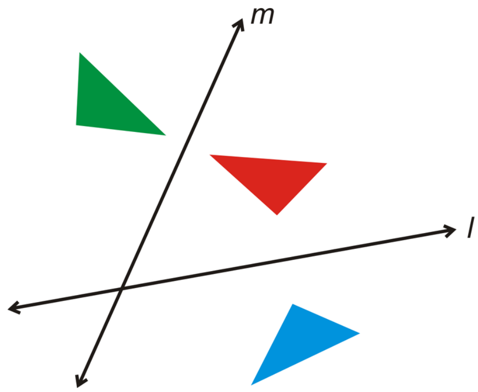
Reflections over Intersecting Lines

Now, we will take the concept we were just discussing and apply it to any pair of intersecting lines. For this composition, we are going to take it out of the coordinate plane. Then, we will apply the idea to a few lines in the coordinate plane, where the point of intersection will always be the origin.

Example 7: Copy the figure below and reflect it over l, followed by m.



Solution: The easiest way to reflect the triangle is to fold your paper on each line of reflection and draw the image. It should look like this:

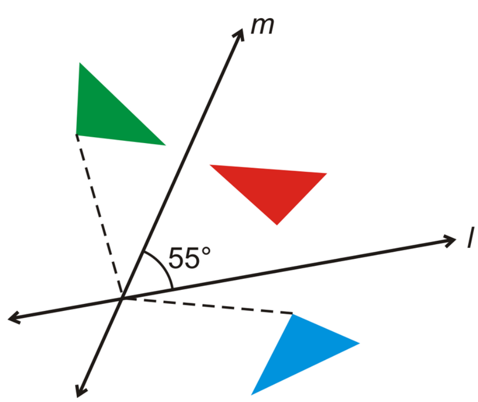


The green triangle would be the final answer.

Investigation 12-2: Double Reflection over Intersecting Lines

Tools Needed: Example 7, protractor, ruler, pencil

a. Take your answer from Example 7 and measure the angle of intersection for lines l and m. If you copied it exactly from the text, it should be about 55°.
b. Draw lines from two corresponding points on the blue triangle and the green triangle. These are the dotted lines in the diagram below.
c. Measure this angle using your protractor. How does it related to 55°?



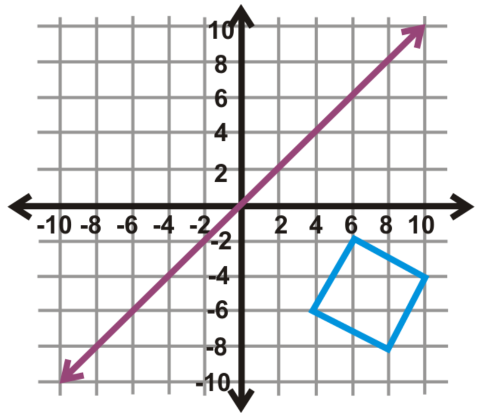
Again, if you copied the image exactly from the text, the angle should be 110°.

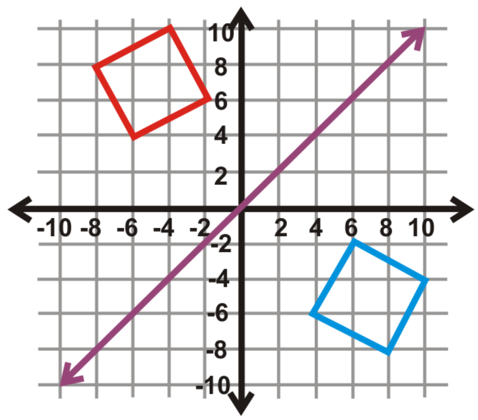
From this investigation, we see that the double reflection over two lines that intersect at a 55° angle is the same as a rotation of 110° counterclockwise, where the point of intersection is the center of rotation. Notice that order would matter in this composition. If we had reflected the blue triangle over m followed by l, then the green triangle would be rotated 110° clockwise.

Reflection over Intersecting Lines Theorem: If you compose two reflections over lines that intersect at x°, then the resulting image is a rotation of $2x^\circ$, where the center of rotation is the point of intersection.

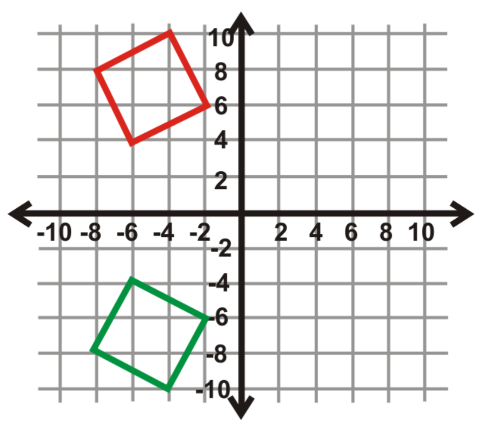
Notice that the Reflection over the Axes Theorem is a specific case of this one.

Example 8: Reflect the square over $y = x$, followed by a reflection over the $x-$axis.



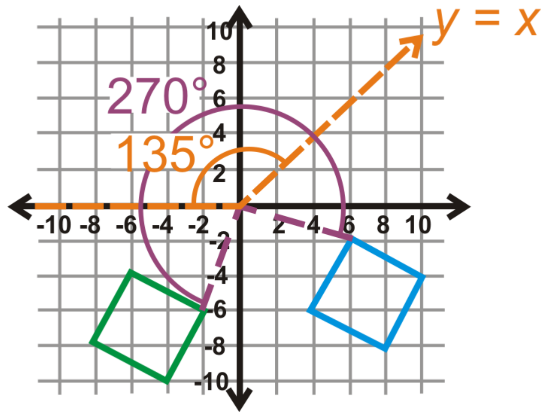


Solution: First, reflect the square over $y = x$. The answer is the red square in the graph above. Second, reflect the red square over the $x-$axis. The answer is the green square below.



Example 9: Determine the one rotation that is the same as the double reflection from Example 8.

Solution: Let's use the theorem above. First, we need to figure out what the angle of intersection is for $y = x$ and the $x-$axis. $y = x$ is halfway between the two axes, which are perpendicular, so is 45° from the $x-$axis. Therefore, the angle of rotation is 90° clockwise or 270° counterclockwise. The correct answer is 270° counterclockwise because we always measure angle of rotation in the coordinate plane in a counterclockwise direction. From the diagram, we could have also said the two lines are 135° apart, which is supplementary to 45°.

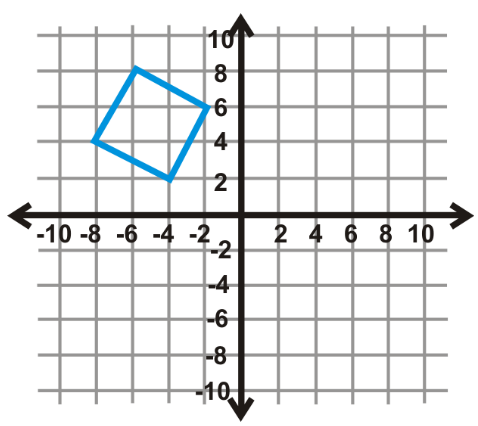


Know What? Revisited The average 6 foot tall man has a $0.415 \times 6 = 2.5$ foot stride. Therefore, the transformation rule for this person would be $(x, y) \rightarrow (-x, y + 2.5)$.

Review Questions

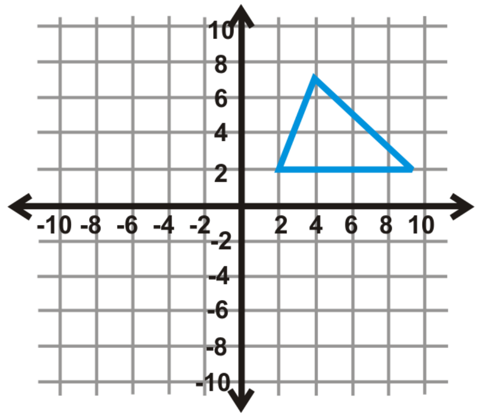
1. *Explain* why the composition of two or more isometries must also be an isometry.
2. What one transformation is equivalent to a reflection over two parallel lines?
3. What one transformation is equivalent to a reflection over two intersecting lines?

Use the graph of the square below to answer questions 4-7.



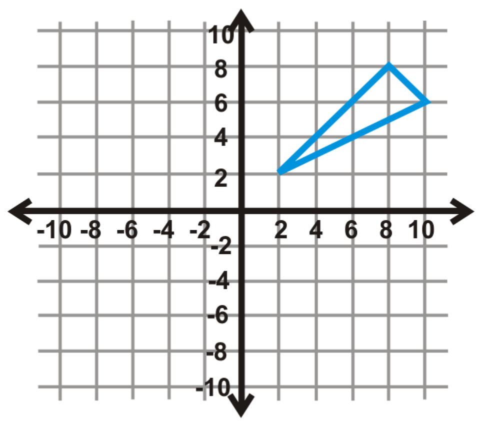
4. Perform a glide reflection over the $x-$axis and to the right 6 units. Write the new coordinates.
5. What is the rule for this glide reflection?
6. What glide reflection would move the image back to the preimage?
7. Start over. Would the coordinates of a glide reflection where you move the square 6 units to the right and then reflect over the $x-$axis be any different than #4? Why or why not?

Use the graph of the triangle below to answer questions 8-10.



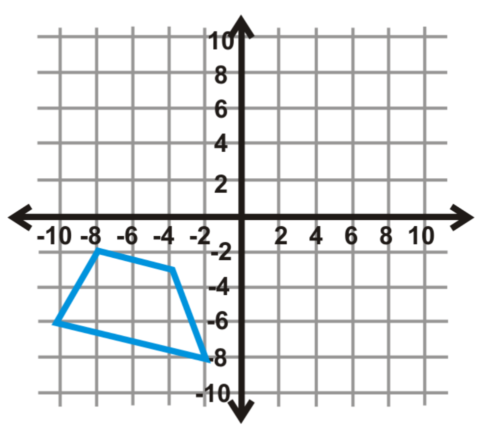
8. Perform a glide reflection over the $y-$axis and down 5 units. Write the new coordinates.
9. What is the rule for this glide reflection?
10. What glide reflection would move the image back to the preimage?

Use the graph of the triangle below to answer questions 11-15.



11. Reflect the preimage over $y = -1$ followed by $y = -7$. Write the new coordinates.
12. What one transformation is this double reflection the same as?
13. What one translation would move the image back to the preimage?
14. Start over. Reflect the preimage over $y = -7$, then $y = -1$. How is this different from #11?
15. Write the rules for #11 and #14. How do they differ?

Use the graph of the trapezoid below to answer questions 16-20.



16. Reflect the preimage over $y = -x$ then the $y-$axis. Write the new coordinates.
17. What one transformation is this double reflection the same as?
18. What one transformation would move the image back to the preimage?
19. Start over. Reflect the preimage over the $y-$axis, then $y = -x$. How is this different from #16?
20. Write the rules for #16 and #19. How do they differ?

Fill in the blanks or answer the questions below.

21. Two parallel lines are 7 units apart. If you reflect a figure over both how far apart with the preimage and final image be?
22. After a double reflection over parallel lines, a preimage and its image are 28 units apart. How far apart are the parallel lines?
23. A double reflection over the x and y axes is the same as a _________ of ________ $^\circ$.
24. What is the center of rotation for #23?
25. Two lines intersect at an 83° angle. If a figure is reflected over both lines, how far apart will the preimage and image be?
26. A preimage and its image are 244° apart. If the preimage was reflected over two intersected lines, at what angle did they intersect?
27. A rotation of 45° clockwise is the same as a rotation of ________ $^\circ$ counterclockwise.
28. After a double reflection over parallel lines, a preimage and its image are 62 units apart. How far apart are the parallel lines?
29. A figure is to the left of $x = a$. If it is reflected over $x = a$ followed by $x = b$ and $b > a$, then the preimage and image are _________ units apart and the image is to the _________ of the preimage.
30. A figure is to the left of $x = a$. If it is reflected over $x = b$ followed by $x = a$ and $b > a$, then the preimage and image are _________ units apart and the image is to the _________ of the preimage.

Review Queue Answers

a. $A'(-2,-8), B'(4,-5), C'(-4,-1), D'(-6,-6)$
b. $A''(2,-8), B''(8,-5), C''(0,-1), D''(-2,-6)$
c. $A'(2,8), B'(8,5), C''(0,1), D''(-2,6)$
d. The coordinates are the same as #2.

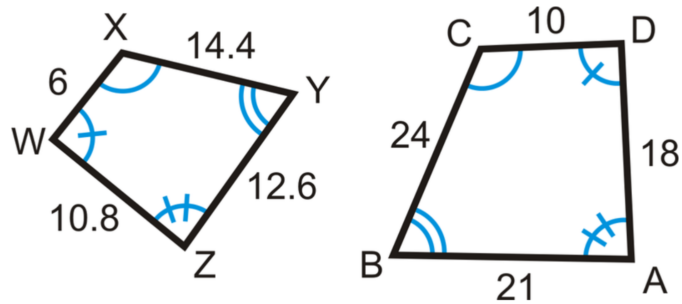
9.6 Dilations

Learning Objectives

- Draw a dilation of a given figure.
- Plot an image when given the center of dilation and scale factor.
- Determine if one figure is the dilation of another.

Review Queue

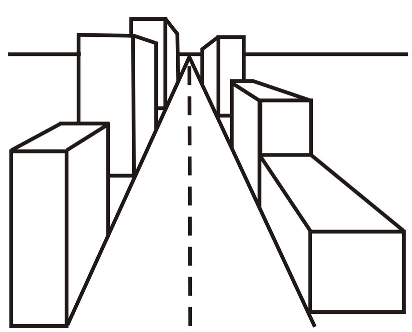
a. Are the two quadrilaterals similar? How do you know?



b. What is the scale factor from $XYZW$ to $CDAB$? *Leave as a fraction.*

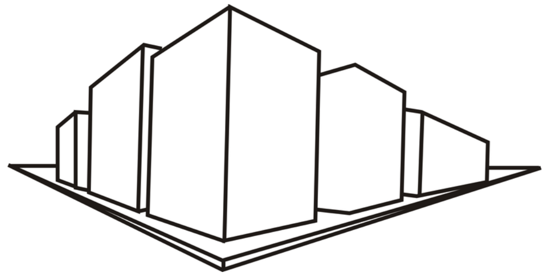
c. Quadrilateral $EFGH$ has vertices $E(-4,-2), F(2,8), G(6,2)$ and $H(0,-4)$. Quadrilateral $LMNO$ has vertices $L(-2,-1), M(1,4), N(3,1)$, and $O(0,-2)$. Determine if the two quadrilaterals are similar. Explain your reasoning.

Know What? One practical application of dilations is perspective drawings. These drawings use a *vanishing point* (the point where the road meets the horizon) to trick the eye into thinking the picture is three-dimensional. The picture to the right is a one-point perspective and is typically used to draw streets, train tracks, rivers or anything else that is linear.



There are also two-point perspective drawings, which are very often used to draw a street corner or a scale drawing of a building.

Both of these drawing are simple representations of one and two perspective drawings. Your task for this **Know What?** is to draw your own perspective drawing with either one or two vanishing points and at least 5 objects. Each object should have detail (windows, doors, sign, stairs, etc.)



Dilations

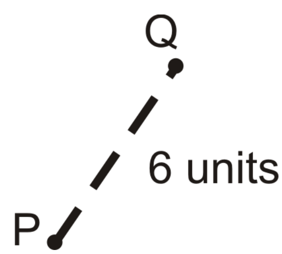
A dilation makes a figure larger or smaller, but has the same shape as the original. In other words, the dilation is similar to the original.

Transformation: An operation that moves, flips, or changes a figure to create a new figure. Transformations that preserve size are ***rigid*** and ones that do not are ***non-rigid.***

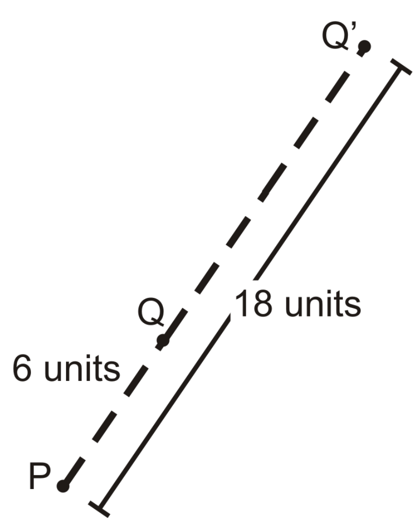
Dilation: A non-rigid transformation that preserves shape but not size.

All dilations have a **center** and a **scale factor.** The center is the point of reference for the dilation (like the vanishing point in a perspective drawing) and scale factor tells us how much the figure stretches or shrinks. A scale factor is typically labeled k and is always greater than zero. Also, if the original figure is labeled $\triangle ABC$, for example, the dilation would be $\triangle ABC$. The ' indicates that it is a copy. This tic mark is said "prime," so A is read "A prime." A second dilation would be A, read "A double-prime."

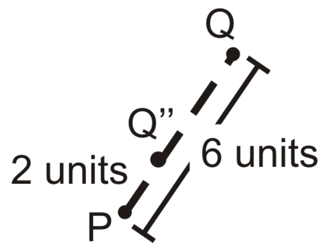
Example 1: The center of dilation is P and the scale factor is 3. Find Q.



Solution: If the scale factor is 3 and Q is 6 units away from P, then Q is going to be $6 \times 3 = 18$ units away from P. Because we are only dilating apoint, the dilation will be collinear with the original and center.

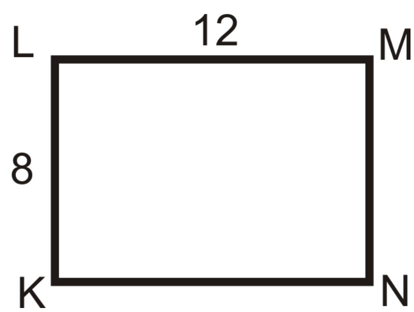


Example 2: Using the picture above, change the scale factor to $\frac{1}{3}$. Find Q.

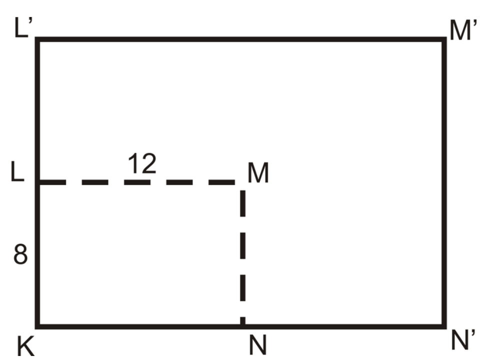


Solution: Now the scale factor is $\frac{1}{3}$, so Q is going to be $\frac{1}{3}$ the distance away from P as Q is. In other words, Q is going to be $6 \times \frac{1}{3} = 2$ units away from P. Q will also be collinear with Q and center.

Example 3: $KLMN$ is a rectangle with length 12 and width 8. If the center of dilation is K with a scale factor of 2, draw $KLMN$.



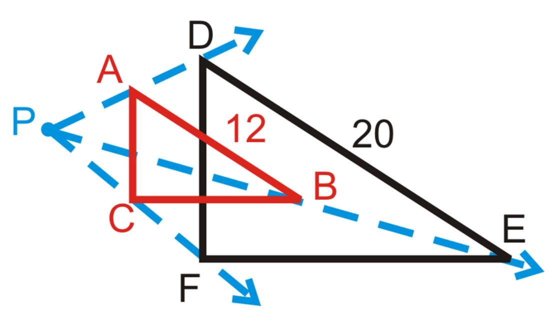
Solution: If K is the center of dilation, then K and K will be the same point. From there, L will be 8 units above L and N will be 12 units to the right of N.



Example 4: Find the perimeters of $KLMN$ and $KLMN$. Compare this to the scale factor.

Solution: The perimeter of $KLMN = 12+8+12+8 = 40$. The perimeter of $KLMN = 24+16+24+16 = 80$. The ratio of the perimeters is 80:40 or 2:1, which is the same as the scale factor.

Example 5: $\triangle ABC$ is a dilation of $\triangle DEF$. If P is the center of dilation, what is the scale factor?



Solution: Because $\triangle ABC$ is a dilation of $\triangle DEF$, we know that the triangles are similar. Therefore the scale factor is the ratio of the sides. Since $\triangle ABC$ is smaller than the original, $\triangle DEF$, the scale factor is going to be a fraction less than one, $\frac{12}{20} = \frac{3}{5}$.

If $\triangle DEF$ was the dilated image, the scale factor would have been $\frac{5}{3}$.

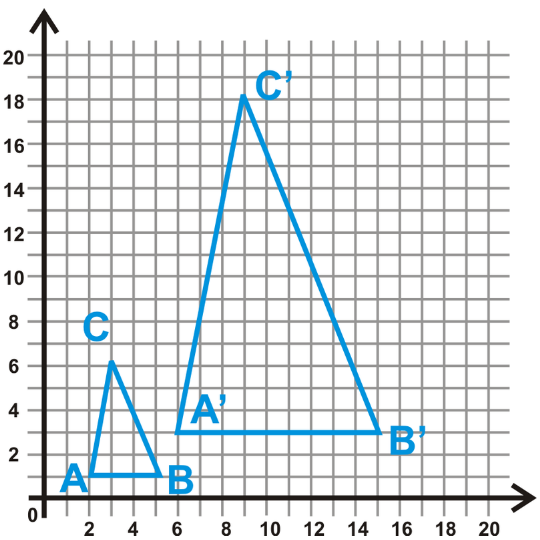
If the dilated image is smaller than the original, then the scale factor is $0 < k < 1$**.**

If the dilated image is larger than the original, then the scale factor is $k > 1$**.**

Dilations in the Coordinate Plane

In this text, the center of dilation will always be the origin, unless otherwise stated.

Example 6: Determine the coordinates of $\triangle ABC$ and $\triangle ABC$ and find the scale factor.



Solution: The coordinates of $\triangle ABC$ are $A(2,1), B(5,1)$ and $C(3,6)$. The coordinates of $\triangle ABC$ are $A(6,3), B(15,3)$ and $C(9,18)$. By looking at the corresponding coordinates, each is three times the original. That means $k = 3$.

Again, the center, original point, and dilated point are collinear. Therefore, you can draw a ray from the origin to C, B, and A such that the rays pass through C, B, and A, respectively.

Let's show that dilations are a similarity transformation (preserves shape). Using the distance formula, we will find the lengths of the sides of both triangles in Example 6 to demonstrate this.

$\underline{\triangle ABC}$	$\underline{\triangle A'B'C'}$
$AB = \sqrt{(2-5)^2+(1-1)^2} = \sqrt{9} = 3$	$A'B' = \sqrt{(6-15)^2+(3-3)^2} = \sqrt{81} = 9$
$AC = \sqrt{(2-3)^2+(1-6)^2} = \sqrt{26}$	$A'C' = \sqrt{(6-9)^2+(3-18)^2} = \sqrt{234} = 3\sqrt{26}$
$CB = \sqrt{(3-5)^2+(6-1)^2} = \sqrt{29}$	$C'B' = \sqrt{(9-15)^2+(18-3)^2} = \sqrt{261} = 3\sqrt{29}$

From this, we also see that all the sides of $\triangle A'B'C'$ are three times larger than $\triangle ABC$. Therefore, ***a dilation will always produce a similar shape to the original***.

In the coordinate plane, we say that A' is a "mapping" of A. So, if the scale factor is 3, then $A(2,1)$ is mapped to (usually drawn with an arrow) $A'(6,3)$. The entire mapping of $\triangle ABC$ can be written $(x,y) \rightarrow (3x,3y)$ because $k = 3$. ***For any dilation the mapping will be*** $(x,y) \rightarrow (kx,ky)$**.**

Know What? Revisited Answers to this project will vary depending on what you decide to draw. Make sure that you have at least five objects with some sort of detail. If you are having trouble getting started, go to the website: http://www.drawing-and-painting-techniques.com/drawing-perspective.html

Review Questions

Given A and the scale factor, determine the coordinates of the dilated point, A'. You may assume the center of dilation is the origin.

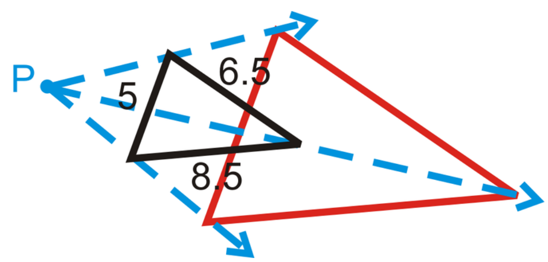
1. $A(3,9), k = \frac{2}{3}$
2. $A(-4,6), k = 2$
3. $A(9,-13), k = \frac{1}{2}$

Given A and A', find the scale factor. You may assume the center of dilation is the origin.

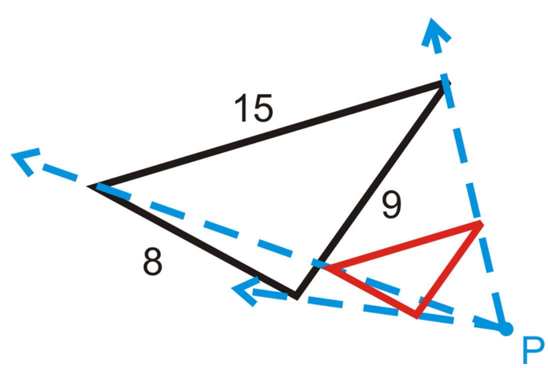
4. $A(8,2), A'(12,3)$
5. $A(-5,-9), A'(-45,-81)$
6. $A(22,-7), A(11,-3.5)$

In the two questions below, you are told the scale factor. Determine the dimensions of the dilation. In each diagram, the **black** figure is the original and P is the center of dilation.

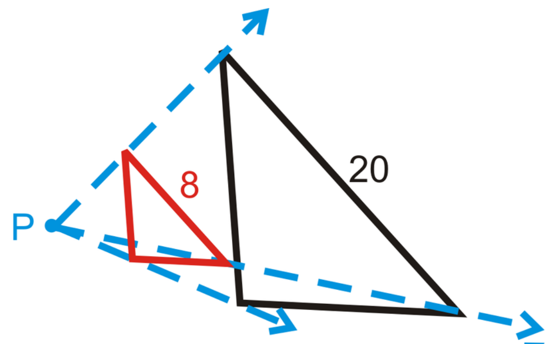
7. $k = 4$

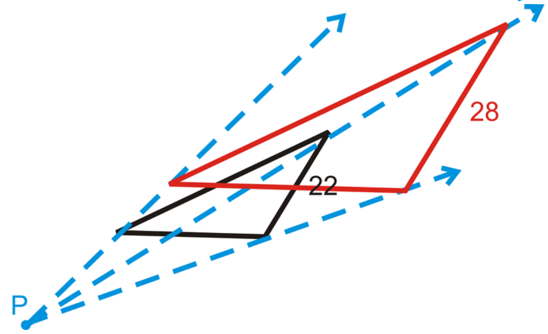


8. $k = \frac{1}{3}$



In the two questions below, find the scale factor, given the corresponding sides. In each diagram, the **black** figure is the original and P is the center of dilation.

9. 

10. 

11. Find the perimeter of both triangles in #7. What is the ratio of the perimeters?
12. ***Writing*** What happens if $k = 1$?

The origin is the center of dilation. Find the coordinates of the dilation of each figure, given the scale factor.

13. $A(2,4), B(-3,7), C(-1,-2); k = 3$
14. $A(12,8), B(-4,-16), C(0,10); k = \frac{3}{4}$

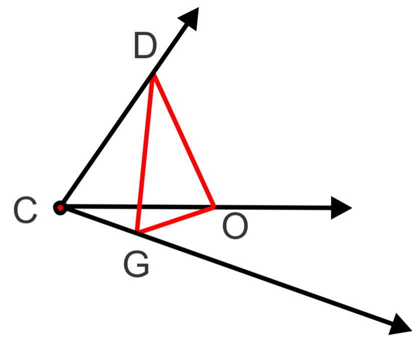
Multi-Step Problem Questions 15-21 build upon each other.

15. Plot $A(1,2), B(12,4), C(10,10)$. Connect to form a triangle.
16. Make the origin the center of dilation. Draw 4 rays from the origin to each point from #15. Then, plot $A'(2,4), B'(24,8), C'(20,20)$. What is the scale factor?
17. Use $k = 4$, to find $A''B''C''$. Plot these points.
18. What is the scale factor from $A'B'C'$ to $A''B''C''$?
19. Find ($O$is the origin):
    a. OA
    b. AA'
    c. AA''
    d. OA'
    e. OA''
20. Find:
    a. AB
    b. $A'B'$
    c. $A''B''$
21. Compare the ratios:
    a. $OA : OA'$ and $AB : A'B'$
    b. $OA : OA''$ and $AB : A''B''$

Algebra Connection For questions 22-27, use quadrilateral $ABCD$ with $A(1,5), B(2,6), C(3,3)$ and $D(1,3)$ and its transformation $A'B'C'D'$ with $A'(-3,1), B'(0,4), C'(3,-5)$ and $D'(-3,-5)$.

22. Plot the two quadrilaterals in the coordinate plane.
23. Find the equation of $\overleftrightarrow{CC'}$.
24. Find the equation of $\overleftrightarrow{DD'}$.
25. Find the intersection of these two lines algebraically or graphically.
26. What is the significance of this point?
27. What is the scale factor of the dilation?

Construction We can use a compass and straight edge to construct a dilation as well. Copy the diagram below.



28. Set your compass to be CG and use this setting to mark off a point 3 times as far from C as G is. Label this point G'. Repeat this process for CO and CD to find O' and D'.
29. Connect G', O' and D' to make $\triangle D'O'G'$. Find the ratios, $\frac{D'O'}{DO}$, $\frac{O'G'}{OG}$ and $\frac{G'D'}{GD}$.
30. What is the scale factor of this dilation?
31. Describe how you would dilate the figure by a scale factor of 4.
32. Describe how you would dilate the figure by a scale factor of $\frac{1}{2}$.

Review Queue Answers

a. Yes, all the angles are congruent and the corresponding sides are in the same ratio.
b. $\frac{5}{3}$
c. Yes, $LMNO \sim EFGH$ because $LMNO$ is exactly half of $EFGH$.

9.7 Tessellations

Learning Objectives

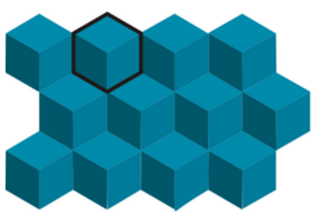
- Determine whether or not a given shape will tessellate.
- Draw your own tessellation.

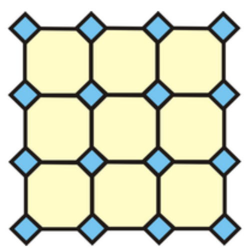
What is a Tessellation?

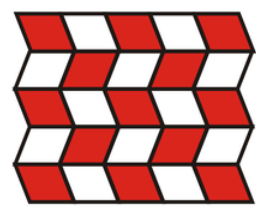
You have probably seen tessellations before, even though you did not call them that. Examples of a tessellation are: a tile floor, a brick or block wall, a checker or chess board, and a fabric pattern.

Tessellation: A tiling over a plane with one or more figures such that the figures fill the plane with no overlaps and no gaps.

Here are a few examples.



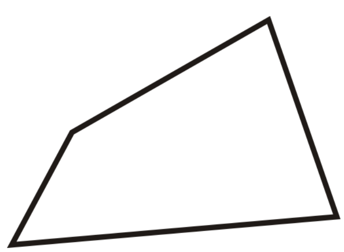




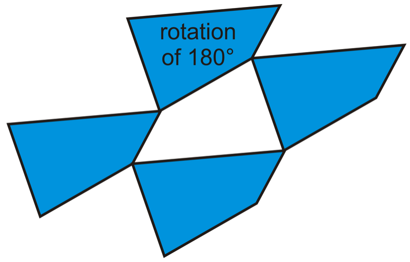
Notice the hexagon (cubes, first tessellation) and the quadrilaterals fit together perfectly. If we keep adding more, they will entirely cover the plane with no gaps or overlaps. The tessellation pattern could be colored creatively to make interesting and/or attractive patterns.

To tessellate a shape it must be able to exactly surround a point, or the sum of the angles around each point in a tessellation must be 360°. Therefore, every quadrilateral and hexagon will tessellate.

Example 1: Tessellate the quadrilateral below.

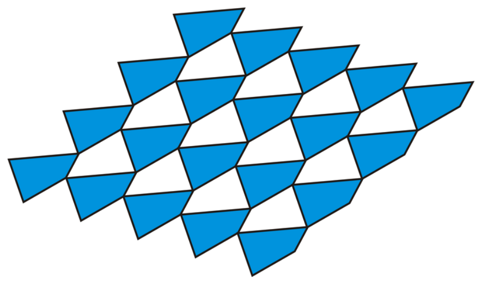


Solution: To tessellate any image you will need to reflect and rotate the image so that the sides all fit together. First, start by matching up each side with itself around the quadrilateral.



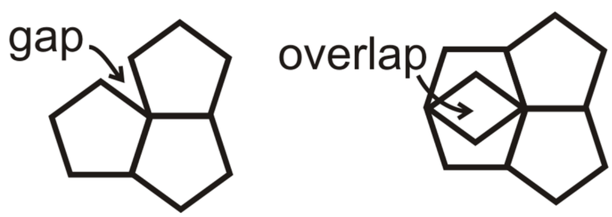
This is the final tessellation. You can continue to tessellate this shape forever.

Now, continue to fill in around the figures with either the original or the rotation.

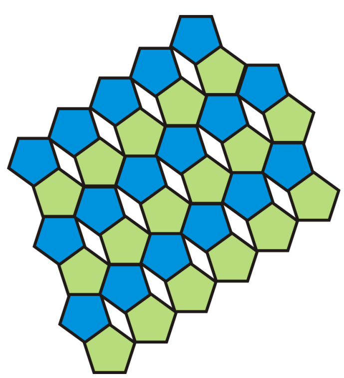


Example 2: Does a regular pentagon tessellate?

Solution: First, recall that there are $(5-2)180^\circ = 540^\circ$ in a pentagon and each angle is $540^\circ \div 5 = 108^\circ$. From this, we know that a regular pentagon will not tessellate by itself because $108^\circ \times 3 = 324^\circ$ and $108^\circ \times 4 = 432^\circ$.



For a shape to be tessellated, the angles around every point must add up to 360°. A regular pentagon does not tessellate by itself. But, if we add in another shape, a rhombus, for example, then the two shapes together will tessellate.



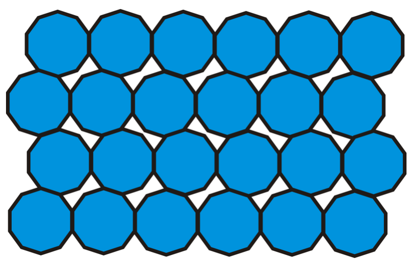
Tessellations can also be much more complicated. Here are a couple of examples.

Review Questions

Will the given shapes tessellate? If so, make a small drawing on grid paper to show the tessellation.

1. A square
2. A rectangle
3. A rhombus
4. A parallelogram
5. A trapezoid
6. A kite
7. A completely irregular quadrilateral
8. Which regular polygons will tessellate?
9. Use equilateral triangles and regular hexagons to draw a tessellation.
10. The blue shapes are regular octagons. Determine what type of polygon the white shapes are. Be as specific as you can.



11. Draw a tessellation using regular hexagons.
12. Draw a tessellation using octagons and squares.
13. Make a tessellation of an irregular quadrilateral using the directions from Example 1.

9.8 Chapter 9 Review

Keywords Theorems

Line of Symmetry
A line that passes through a figure such that it splits the figure into two congruent halves.

Line Symmetry
When a figure has one or more lines of symmetry.

Rotational Symmetry
When a figure can be rotated (less that 360°) and it looks the same way it did before the rotation.

Center of Rotation
The point at which the figure is rotated around such that the rotational symmetry holds. Typically, the center of rotation is the center of the figure.

angle of rotation
The angle of rotation, tells us how many degrees we can rotate a figure so that it still looks the same.

Transformation
An operation that moves, flips, or changes a figure to create a new figure.

Rigid Transformation
A transformation that preserves size and shape.

Translation
A transformation that moves every point in a figure the same distance in the same direction.

Vector
A quantity that has direction and size.

Reflection
A transformation that turns a figure into its mirror image by flipping it over a line.

Line of Reflection
The line that a figure is reflected over.

Reflection over the $y-$axis
If (x,y) is reflected over the $y-$axis, then the image is $(-x,y)$.

Reflection over the $x-$axis
If (x,y) is reflected over the $x-$axis, then the image is $(x,-y)$.

Reflection over $x=a$
If (x,y) is reflected over the vertical line $x=a$, then the image is $(2a-x,y)$.

Reflection over $y = b$

If (x,y) is reflected over the horizontal line $y = b$, then the image is $(x, 2b - y)$.

Reflection over $y = x$

If (x,y) is reflected over the line $y = x$, then the image is (y,x).

Reflection over $y = -x$

If (x,y) is reflected over the line $y = -x$, then the image is $(-y,-x)$.

Rotation

A transformation by which a figure is turned around a fixed point to create an image.

Center of Rotation

The fixed point that a figure is rotated around.

Rotation of 180°

If (x,y) is rotated 180° around the origin, then the image will be $(-x,-y)$.

Rotation of 90°

If (x,y) is rotated 90° around the origin, then the image will be $(-y,x)$.

Rotation of 270°

If (x,y) is rotated 270° around the origin, then the image will be $(y,-x)$.

Composition (of transformations)

To perform more than one rigid transformation on a figure.

Glide Reflection

A composition of a reflection and a translation. The translation is in a direction parallel to the line of reflection.

Reflections over Parallel Lines Theorem

If you compose two reflections over parallel lines that are h units apart, it is the same as a single translation of $2h$ units.

Reflection over the Axes Theorem

If you compose two reflections over each axis, then the final image is a rotation of 180° of the original.

Reflection over Intersecting Lines Theorem

If you compose two reflections over lines that intersect at x°, then the resulting image is a rotation of $2x^\circ$, where the center of rotation is the point of intersection.

Tessellation

A tiling over a plane with one or more figures such that the figures fill the plane with no overlaps and no gaps.

Review Questions

Match the description with its rule.

1. Reflection over the $y-$axis - A. $(2a-x,y)$
2. Reflection over the $x-$axis - B. $(-y,-x)$
3. Reflection over $x=a$ - C. $(-x,y)$
4. Reflection over $y=b$ - D. $(-y,x)$
5. Reflection over $y=x$ - E. $(x,-y)$
6. Reflection over $y=-x$ - F. $(x,2b-y)$
7. Rotation of 180° - G. (x,y)
8. Rotation of 90° - H. $(-x,-y)$
9. Rotation of 270° - I. $(y,-x)$
10. Rotation of 360° - J. (y,x)

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9697 .